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Aids to Computation

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Guide to Tables of Elliptic Functions

by

ALAN FLETCHER

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Note: In the various parts, differences are referred to as in FMR *Index*. In the description of a table, Δ means that first differences only are given; Δ^2 means first and second differences, δ^2 means second differences only, δ_m^2 means modified second differences, Δ^4 means first to fourth differences, δ^4 means second and fourth differences, and so on.

EDITORIAL NOTE

Of material which HARRY BATEMAN had prepared for his Subcommittee L–Report on Higher Mathematical Functions we have already published the part dealing with BESSEL Functions (*MTAC*, no. 7, 1944). Since in varied research problems there seemed now to be an increasing demand for tabular information regarding Elliptic Functions, we invited Dr. FLETCHER, of the University of Liverpool, to prepare a Guide to this material as another part of our Section L–Report,—quite independent of anything H.B. had collected. The following comprehensive survey indicates not only all tables, but also all known errata, including many previously unpublished, discovered in Dr. FLETCHER'S personal investigations. We are indeed grateful for this valuable aid in the promotion of scientific inquiry.

R. C. A.

PREFACE

In the title, "elliptic functions" is to be understood in a general sense, as including elliptic integrals and theta functions; in fact, existing tables, and therefore the present *Guide*, relate largely to elliptic integrals. "Tables" means numerical tables; extensive lists of formulae are given in several of the works quoted, but no account is given here of the enormous amounts of algebraical calculation which have been devoted to transformations

of order higher than the third. The outlook is practical, and in sympathy with LUCIEN LÉVY's dictum that "the most elementary properties of the elliptic functions are the most useful."

It will be observed that the *Guide* is divided into the following three parts: Part I, a descriptive *Guide* to all tables considered, referring, with constant obvious abbreviations, to material listed in the detailed bibliography of Part II; and Part III, devoted to lists of errata in certain tables marked with an asterisk in Part II.

There is naturally in Part I an appreciable amount of duplication with FMR *Index* regarding material published before 1946; but in the present Part II much greater detail is offered, and Part III is almost entirely new.

Great care has been taken in Part III to indicate the nature of the verifications made and the status of the error lists given. A mere list of known errors, unaccompanied by any statement regarding its degree of completeness, is of limited use to the computer. Unless it can be asserted that the table in question contains no other gross errors, or at any rate no errors exceeding a definite size, he is still in the position of not having complete confidence in any entry. In several cases I have stated that a table is free from gross error (or free from any error); such a statement is more valuable than an imposing but uncharacterized list of errors. In a few cases where I have not examined a table, but merely communicate a few errors found by others, I have been careful to make clear the limited nature of the information given.

For information on the numerical calculation of elliptic integrals and functions, one may consult, among other references, L. V. KING, *On the Direct Numerical Calculation of Elliptic Functions and Integrals*, Cambridge, 1924, and a number of papers by S. C. VAN VEEN in Akad. Wetens., Amsterdam, *Proc.*, v. 44–45, 1941–42.

In performing about half the differencing which was done on the National accounting machine in the Mathematical Laboratory at Liverpool I had the invaluable assistance of Miss OLIVE E. VANES. I also wish to thank various library officials, friends and colleagues, and the University of Liverpool, for encouragement and arrangement of facilities for carrying on my investigations; and Professor R. C. ARCHIBALD for his encouragement and his care in editing the manuscript.

PART I

GUIDE TO TABLES

Section I: TABLES CONCERNING THE MODULUS

The description of tables of elliptic integrals and functions is complicated by the fact, which will become very noticeable, that various ways of specifying the modulus k have been used, so that the various tables employ several different argument-systems. The older use of the modular angle $\sin^{-1} k$ (which we shall denote by θ, except, for a special reason, in Section IX) and the more recent use of k^2 are by far the most important for general mathematical purposes, since they facilitate tabulation of functions of the complementary modulus k' defined by $k' = \sqrt{(1 - k^2)} = \cos\theta$ (k and k' will almost always be real and between 0 and 1); but other arguments (k, k', $\log k$, $\log k'^2$, $\tan^2\theta$, etc.) are convenient for special purposes.

Several tables for conversion of system (tables of θ against k^2, k^2 against θ, and k against k^2) are given in HAYASHI 1, 2. These tables contain a number of errors. In most cases a computer, presumed to be using a calculating machine, will do best to perform his own squarings and extractions of square root, when a reliable table, such as BARLOW-COMRIE, does not suffice; while the standard trigonometric and logarithmic tables are usually to be preferred, because of their great extent and accuracy, to any small table offered among elliptic tabulations.

Two tables for applying LANDEN's transformation will now be mentioned. The formulae for decreasing the modulus from k to k_1, so making $K_1'/K_1 = 2K'/K$, that is, doubling the

period ratio, are

$$k_1 = (1 - k')/(1 + k') < k, \qquad k_1' = 2\sqrt{k'}/(1 + k') > k'.$$

$k_1'^2$ is tabulated to 7D with first differences for $k'^2 = 0(.001)1$ in NAGAOKA & SAKURAI 1 (p. 49). Successive applications of Landen's transformation provide a method of computing elliptic integrals, and 14D logarithms of the sequence of moduli and complementary moduli are given in LEGENDRE 3 (T.VI), 5 (T.VI) for $\theta = 0(0°.1)15°(0°.5)45°$.

Elliptic calculations are frequently made by methods involving the evaluation of arithmetic-geometric means, usually between 1 and k'. The process is a variant form of repeated Landen transformation. Usually only its end-result is of interest. But on account of their extent, it may be noticed that the first six sets of number triplets are given to 17D for $k^2 = 0(.01)1$ in the unpublished NBSCL 2. See also the description of BARTKY 1 in Section IX.

LEGENDRE 4 (p. 226) gives a table of modular angles in the cubic transformation. If the modular angles are θ and λ ($\theta < \lambda$), and the corresponding quarter-periods are K, K', Δ, Δ', then $K'/K = 3\Delta'/\Delta$, provided that

$$p = \sin(M - 30°)/\cos M, \quad q = \cos(M - 30°)/\sin M, \quad 30° \leqslant M \leqslant 60°,$$
$$\sin^2\theta = p^2q, \qquad \sin^2\lambda = pq^2.$$

LEGENDRE 4 gives θ and λ to 5D of 1° for $M = 30°(1°)60°$. At $M = 45°$ we have the well-known case $\theta = 15°$, $\lambda = 75°$; moreover the second half of the table may be written down from the first, by changing M, θ, λ into $90° - M$, $90° - \lambda$, $90° - \theta$ respectively. For a related table, see F. W. NEWMAN 1, 2 in Section XI.

Section II: COMPLETE ELLIPTIC INTEGRALS OF THE FIRST AND SECOND KINDS

The incomplete elliptic integrals of the first and second kinds will be denoted by, respectively,

$$F(\phi) = \int_0^\phi (1 - k^2\sin^2\phi)^{-\frac{1}{2}}d\phi \qquad \text{and} \qquad E(\phi) = \int_0^\phi (1 - k^2\sin^2\phi)^{\frac{1}{2}}d\phi.$$

These are functions of two variables, namely ϕ and k (or $\sin^{-1}k$, k^2, etc.). In this section we are concerned only with the corresponding complete integrals in which the upper limit is $\frac{1}{2}\pi$, which we shall denote as usual by

$$K = \int_0^{\frac{1}{2}\pi} (1 - k^2\sin^2\phi)^{-\frac{1}{2}}d\phi \qquad \text{and} \qquad E = \int_0^{\frac{1}{2}\pi} (1 - k^2\sin^2\phi)^{\frac{1}{2}}d\phi.$$

These are functions of one variable, indicating the modulus, only, and are what is most often required in applications of elliptic integrals. Along with tables of K, E and their logarithms, we consider tables of $2K/\pi$, $2E/\pi$ and $\pi/2K$; the last of these is equal to M, the arithmetic-geometric mean of 1 and k'. Sections III–VII will be concerned with further expressions involving K, E and the corresponding functions K', E' of the complementary modulus.

As k tends to unity, there is apt to be difficulty in interpolating a value of K (which tends to infinity), and to a smaller extent one of E. Auxiliary tables for the calculation of K and E in such a case have been given by various authors. These are duly listed in E below. Attention is particularly drawn to the recent table of KAPLAN 1 (listed in A_3 below), which tabulates K and E to 10D with argument $\log k'^2$. It is interesting to find that, with this argument, the second differences of K become less than those of E. Towards the end of KAPLAN's table, in the region where interpolation in ordinary tables is practically impossible, interpolation is linear to 9D in both K and E.

The lists at the end of this Section refer to tables in which k^2 is real. But diagrams for K and E when k^2 is complex are given in JAHNKE & EMDE 1_2–1_4.

The tables of CAMBI 1 tabulate complete integrals for moduli given by

$$k^2 = -e^{iy} = e^{-iy}, \qquad k'^2 = 1 + e^{iy} = 1 - e^{-iy},$$

where $y = \pi - x$. For $x = 0(\pi/20)\pi$, with additional arguments when necessary near

$x = \pi$ ($k^2 = 1$), the tables give, to 7–10D, values of

$$k', K, K', K/K', -\log|q'|, k'K \text{ and } e^{k'\pi}(1 - E'/K').$$

On the circle of tabulation, $q' = e^{-\pi K/K'}$ is purely imaginary. For fuller details, see RMT 565.

II-A$_1$. *K and E, argument* θ

15D, SPENCELEY 1, $\theta = 0(1°)90°$
12D, LEGENDRE 3 (T.VIII), 5 (T.VIII), 7, $\theta = 0(1°)90°$, Δ^5 to 45°, then Δ
12D, MOSELEY 1, $\theta = 0(1°)90°$, from LEGENDRE
12D, *K* only, F. W. NEWMAN 2 (p. 29, 132), $\theta = 0(1°)90°$, from LEGENDRE
9–10D, LEGENDRE 3 (T.IX), 5 (T.IX), 6, 7, 8, $\theta = 0(1°)90°$ ($\phi = 90°$ in double-entry tables)
9–10D, HIPPISLEY 1, $\theta = 0(1°)10°(5°)80°(1°)90°$
9D, S.A.E.W.E.B. 1 (p. 345), $\theta = 0(2°.5)90°$
7D, BOLL 1 (p. 351), $\theta = 0(1°)70°(0°.5)80°(0°.2)89°(0°.1)90°$
7D, *K* only, LÁSKA 1, $\theta = 0(1°)90°$
6D, ROSA & COHEN 1 and ROSA & GROVER 1, $\theta = 0(1°)90°$, Δ^2
6D, GRAY 1_2, $\theta = 0(1°)90°$, Δ
6D, RUSSELL 1, $\theta = 0(1°)90°$
5D, KIEPERT 1, $\theta = 0(0°.2)90°$
5D, LÉVY 1, HANCOCK 1, HAYASHI 2, $\theta = 0(1°)70°(0°.5)80°(0°.2)89°(0°.1)90°$
5D, DALE 1, FRICKE 1, $\theta = 0(1°)90°$
5D, *K* only, DWIGHT 2, $\theta = 86°(1')90°$, Δ
4D, SILBERSTEIN 1, $\theta = 0(1°)45°(0°.5)80°(0°.2)89°(0°.1)90°$
4D, PEIRCE 2, BURINGTON 1, ALLEN 1, $\theta = 0(1°)65°(0°.5)80°(0°.2)89°(0°.1)90°$
4D, ROSENBACH, WHITMAN & MOSKOVITZ 1, $\theta = 0(1°)70°(30')80°(12')89°(6')89°30'(3')90°$
4D, JAHNKE & EMDE 1, GLAZENAP 1, $\theta = 0(1°)70°(0°.5)80°(0°.2)89°(0°.1)90°$
4D, LIGOWSKI 1, $\theta = 0(0°.5)89°.5(0°.1)90°$
4D, HODGMAN 1, $\theta = 0(1°)90°$, also *K* to 3D for $\theta = 85°(6')89°(2')89°40'(1')90°$
4D, FOWLE 1, PEIRCE 1, HÜTTE 1, $\theta = 0(1°)90°$
3D, *K*, DWIGHT 1, $\theta = 0(1°)50°(30')70°(12')82°(6')89°(2')89°40'(1')90°$, Δ
3D, *E*, DWIGHT 1, $\theta = 0(1°)90°$, Δ
3D, MONTESSUS DE BALLORE 1, small tables

II-A$_2$. *K and E, argument* k^2

16D, *K* only, NBSCL 2, $k^2 = 0(.01)1$
12D, *K*, HAYASHI 1, $k^2 = .835(.001)1$
10D, *K*, HAYASHI 1, $k^2 = 0(.001).834$, $k^2 = 0(10^{-7})10^{-5}(10^{-5}).00249$
8D, *K*, HAYASHI 1, $k^2 = .00250(10^{-5}).003$, $1 - k^2 = 0(10^{-7})10^{-5}(10^{-5}).003$
10D, *E*, HAYASHI 3, $k^2 = 0(.001)1$
9D, MILNE-THOMSON 2, $k^2 = 0(.01)1$, Δ
7D, SAMOĬLOVA-ĹAKHONTOVA 1, $k^2 = 0(.001)1$, Δ
7D, MILNE-THOMSON 3, $k^2 = 0(.01)1$
6D, NAGAOKA & SAKURAI 1, $k^2 = 0(.001)1$, Δ
5D, HAYASHI 2, $k^2 = 0(.001)1$
5D, *E*, DWIGHT 3, $k^2 = 0(.001)1$, $I\Delta$
4D, *K*, DWIGHT 3, $k^2 = 0(.001).997(10^{-5}).99999(10^{-7})1$, $I\Delta$ in most of range
4D, JAHNKE & EMDE 1_2, 1_4, $k^2 = 0(.01)1$, $I\Delta$

II-A$_3$. *K and E, other arguments*

12D, *K*, FLETCHER 3, $k = 0(.01).7(.005)1$, Δ^n
12D, *E*, FLETCHER 3, $k = 0(.01).9(.005)1$, Δ^n
11D, *E* only, SCHMIDT 1 (p. 204), $k' = 0(.01)1$
10D, KAPLAN 1, $\log k'^2 = -1(.005) - 2(.01) - 6$, $\mu\delta^2$ or $\frac{1}{4}\mu\delta^2$
10D, FLETCHER 2, $k = 0(.01)1$

7D, ROSA & COHEN 1 and ROSA & GROVER 1, $\tan\theta = 0(.1)1(.5)3(1)5(2.5)12.5$
5D, E only, SCHLÖMILCH 1, $k' = 0(.01)1$
4D, $4E$ only, JELÍNEK 1, $k' = 0(.01)1$
4D, $2E$ only, MOORE 1, $k' = 0(.01)1$ [very inaccurate, see Part III]
4D, E, CECCONI 1, $k = 0(.01)1$ [see also Section VII-4]
3D, E only, WAYNE 1, $k' = 0(.01)1$

II-B. *Log K and log E*

12D, LEGENDRE 3 (T.I), 5 (T.I), 8, $\theta = 0(0°.1)90°$, Δ^3 to 70°, then Δ^4. (From 0 to 15° and from 75° to 90°, 14D values, not very accurate, are given for the functions, but the differences throughout are those of the 12D values.)
8D, ROSA & COHEN 1 and ROSA & GROVER 1, $\theta = 45°(0°.1)90°$, Δ^2
8D, RUSSELL 1, $\theta = 80°(0°.1)90°$, Δ^2
7D, BERTRAND 1 (p. 714), $\theta = 0(0°.5)90°$, K and E called ω and ζ
7D, GREENHILL 3, POTIN 1, $\theta = 0(0°.5)90°$
7D, LEGENDRE 1 (p. 118), $\theta = 0(1°)90°$
7D, ROSA & COHEN 1 and ROSA & GROVER 1, $\tan\theta = 0(.1)1(.5)3(1)5(2.5)12.5$
6D, HODGMAN 1, $\theta = 0(1°)90°$, also $\log K$ to 5D for $\theta = 85°(6')89°(2')89°40'(1')90°$
6D, FOWLE 1, $\theta = 0(1°)90°$
5D, BOHLIN 1, WITKOWSKI 1, $\theta = 0(0°.1)90°$
4D, HOÜEL 1, $\theta = 0(1^g)100^g$

II-C. *$2K/\pi$, $2E/\pi$, $M = \frac{1}{2}\pi/K$ and their logarithms*

12D, $2K/\pi$, $2(K-E)/\pi$, KAPLAN 2, $\tan^2\theta = -.005(.005) + .160$, δ_m^2
6D, $2K/\pi$, $2E/\pi$, HEUMAN 1, $\theta = 0(0°.1)90°$, Δ
9D, $2K/\pi$, also $(2K/\pi)^2$, S.A.E.W.E.B. 1 (p. 345), $\theta = 0(2°.5)90°$
14D, $\log(2K/\pi)$, LEGENDRE 3 (T.VI), 5 (T.VI), $\theta = 0(0°.1)15°(0°.5)45°$
4D, $\log(2K/\pi)$, HOÜEL 1, $\theta = 0(1^g)100^g$
17D, M, NBSCL 2, $k^2 = 0(.01)1$
11–15D, M, GAUSS 2 (p. 363), $k' = .2, .6, .8$
12D, M, FLETCHER 3, $k = 0(.01).7(.005)1$, Δ^n
10D, M, FLETCHER 2, $k = 0(.01)1$
10D, $M\ (=1/\vartheta_3^2)$, MILNE-THOMSON 2, $k^2 = 0(.01)1$, Δ
12D, M, HAYASHI 1, $k^2 = .835(.001)1$
8D, M, HAYASHI 1, $k^2 = .997(10^{-5}).99999(10^{-7})1$
7D, M, $\log M$, GAUSS 2 (p. 403), $\theta = 0(0°.5)90°$
See also Section V.

II-D. *K'/K, E/K*

10D, K'/K, HAYASHI 1, $k^2 = 0(.001)1$
8D, K'/K, K/K', HAYASHI 1, $k^2 = 0(10^{-7})10^{-5}(10^{-5}).003$
7D, E/K, MILNE-THOMSON 4, $k^2 = 0(.1)1$

II-E. *Auxiliary tables, mainly for fairly large k*

13–12D, K_1, K_2, E_1, E_2, AIREY 1, $k'^2 = 0(.00001).00010$, δ^2, $K = K_1 \ln(4/k') - K_2$, $E = E_1 \ln(4/k') + E_2$
12–11D, K_1, K_2, E_1, E_2, AIREY 1, $k'^2 = 0(.0001).0010$, δ^2
10D, K_1, K_2, AIREY 1, $k'^2 = 0(.001).170$, δ^2
10D, E_1, E_2, AIREY 1, $k'^2 = 0(.001).100$, δ^2
8D, logarithms of C, C', C_1, C_1', C_2, WITT 1, $\theta = 13°(1°)36°(30')45°(20')50°(10')60°$, Δ, where $K = \sec\frac{1}{2}\theta\ (\sec\theta)^{\frac{1}{2}}C$, $E = \cos^2\frac{1}{2}\theta\ (\sec\theta)^{\frac{1}{2}}C_1$, $C' = 2C/\pi$, $C_1' = 2C_1/\pi$, $C_2 = C - C_1$. More complicated functions for $\theta \geq 60°$. See also WITT 2
8D, $\log M$, IDELSON 1, $\theta = 60°(2°)64°(1°)75°$, where $2K/\pi = \sec^2\frac{1}{2}\psi\ (\sec\frac{1}{2}\theta)^{\frac{1}{2}}\ (\sec\theta)^{\frac{1}{2}}\ M$, $\cos\psi = (\cos\theta)^{\frac{1}{2}}$. More complicated function for $\theta > 75°$

8D, $\log (K/2\pi) + 2 \log (1 + \sqrt{k'})$, ROBBINS 2, $k = .4(.01).6(.001).849$
7D, $\log [(4K/\pi) \cos \frac{1}{2}\theta \, (\cos \theta)^{\frac{1}{2}}]$, NEWCOMB 1 (p. 69), $k = .45(.01).75$
7–11D, $K/\ln (4/k')$, DWIGHT 2, $\theta = 86°(1')90°$, Δ
7D, $K/\log (4/k')$, SAMOĬLOVA-ĨAKHONTOVA 1, $k^2 = .95(.001)1$, Δ
6D, $2K/\pi - a \log (90 - \theta)$, $a = (2/\pi) \ln 10 = 1.465871$, HEUMAN 1, $\theta = 65(0.1)90$, Δ
5D, first 8 coefficients in expansions of $K - \frac{1}{2}\pi + \ln k'$ and $K' + (2K/\pi) \ln k - \ln 4$ in powers of k^2, TANNERY & MOLK 1 (v. 3, p. 215)
4D, $K - \ln (4/k')$, $\ln (4/k')$, JAHNKE & EMDE 1_3, 1_4, $k^2 = .7(.01)1$, $I\Delta$
4D, $\beta = \log K - \log \ln (4/k')$, HOÜEL 1 (p. 57), $\theta = 50^g(1^c)100^c$
4D, $4E/\pi(1 + k')$, BOLL 1 (p. 348), $\epsilon = (1 - k')/(1 + k') = 0(.01)1$
4D, $4E/\pi(1 + k')$, BOLL 1 (p. 349), $k' = 0(.01)1$

Section III: JACOBI'S NOME q

q is defined as $e^{-\pi K'/K}$. Thus $\ln q = -\pi K'/K$, or $\log q = -\mu\pi K'/K$, where $\mu\pi = \pi \log e = 1.36437\ 63538\ 41841$, so that a table of $\log q$ is easily computed from a table of K, if θ or k^2 is the argument. Also

$$\log \log (1/q) = \log (\mu\pi) + \log K' - \log K,$$

where

$$\log (\mu\pi) = 0.13493\ 41839\ 94670\ 6,$$

so that $\log \log (1/q)$ is very easily computed from a table of $\log K$. It is not essential that a table of $\log q$ or of $\log \log (1/q)$ should extend beyond $\theta = 45°$ or $k^2 = \frac{1}{2}$, for if q' is the complementary nome $e^{-\pi K/K'}$, we have

$$\log q \log q' = \mu^2\pi^2 = 1.86152\ 28349\ 22757$$

and

$$\log \log (1/q) + \log \log (1/q') = 2 \log (\mu\pi) = 0.26986\ 83679\ 89341\ 3.$$

If $q = \Sigma a_n k^{2n}$, the exact values of a_n were calculated for $n = 1(1)12$ by F. TISSERAND, published in HERMITE 1_1, and reproduced in HERMITE 1_2 and in TANNERY & MOLK 1 (v. 4, p. 121).

The most usual way of computing q, other than by using its definition as given above, is to put

$$2\epsilon = (1 - \sqrt{k'})/(1 + \sqrt{k'}),$$

when we have

$$\epsilon = (q + q^9 + q^{25} + \ldots)/(1 + 2q^4 + 2q^{16} + \ldots),$$

which inverts (see WEIERSTRASS & SCHWARZ 1, p. 56) into

$$q = \epsilon + 2\epsilon^5 + 15\epsilon^9 + 150\epsilon^{13} + \ldots.$$

The first 14 terms of the series are given in LOWAN, BLANCH & HORENSTEIN 1.

It should be noted that q and ϵ are called h and $\frac{1}{2}l$ respectively in WEIERSTRASS & SCHWARZ 1 and elsewhere. The small difference $q - \epsilon$ (called $q - \frac{1}{2}l$) is tabulated to 8D for $q = 0(.01).14$ in NAGAOKA 1, and to 8D with Δ for $q = .02(.002).1(.001).15$ in NAGAOKA 2; the second table is reproduced in ROSA & GROVER 1. NAGAOKA & SAKURAI 1 (p. 56) tabulates $\log 2\epsilon$ (called $\log l$) to 7D with Δ for $k^2 = 0(.001).5$.

For complex k^2, see CAMBI 1 in Section II; also the diagrams in JAHNKE & EMDE 1_7 (p. 120), 1_5–1_6 (p. 46), giving k^2 as a function of τ, where $q = e^{i\pi\tau}$.

III–A. *q and its powers*

16S, q, SPENCELEY 1, $\theta = 0(1°)90°$
14–15D, q, HIPPISLEY 1, $\theta = 0(5°)80°(1°)90°$
8D, q, GLAISHER 1, $\theta = 0(1°)90°$
7–5D, q, LÁSKA 1, $\theta = 0(1°)90°$
17D, q^n, NBSCL 2, $n = \frac{1}{4}, \frac{1}{2}, 1(1)4(2)8, 9, 12, 16, 20, 25$, $k^2 = 0(.01)1$

10D, q, MILNE-THOMSON 1, $k^2 = 0(.01)1$, Δ
8D, q, MILNE-THOMSON 3, $k^2 = 0(.01)1$
8D, q, SAMOĬLOVA-ĨAKHONTOVA 1, $k^2 = 0(.001)1$, Δ
8D, q, HAYASHI 1, $k^2 = 0(.001).5$
5D, q, HAYASHI 2, $k^2 = 0(.001).5$
7D, q, GROVER 6 (p. 251), $k^2 = 0(.005).1$, Δ^2
11D, $q^{25/4}$, HAYASHI 1, $k^2 = .3(.001).5$
10D, q^4, $q^{9/4}$, HAYASHI 1, $k^2 = 0(.001).5$
8D, $q^{1/4}$, HAYASHI 1, $k^2 = 0(.001).5$

III-B. $\pm$ *log q and auxiliary functions*

10D, log $(1/q)$, PLANA 1, $\theta = 0(0°.1)45°(1°)90°$ [highly inaccurate, see Part III]
10D, log q, INNES 2, $\theta = 0(1°)45°$
10D, log $(q \cot^2 \frac{1}{2}\theta)$, INNES 2, $\theta = 0(1°)45°$, log v^4
10D, log (q/ϵ), INNES 2, $\theta = 0(1°)31°(30')40°(10')45°$
8D, log q, MEISSEL 1, $\theta = 0(1')90°$
8D, log q, GLAISHER 1, $\theta = 0(1°)90°$
5D, log q, JACOBI 1, $\theta = 0(0°.1)90°$, Δ
5D, log q, BERTRAND 1, LÉVY 1, POTIN 1, $\theta = 0(5')90°$
5D, log q, FRICKE 1, $\theta = 0(10')90°$
5D, log q, SCHLÖMILCH 2, SILBERSTEIN 1, $\theta = 0(1°)90°$
5D, log $(1/q)$, LÁSKA 1, $\theta = 0(1°)90°$
4D, log q, JAHNKE & EMDE 1, $\theta = 0(5')90°$
4D, log q, BOHLIN 1, GLAZENAP 1, $\theta = 0(0°.1)90°$
4D, log q, HOÜEL 1, $\theta = 0(1^g)100^g$
4D, $\alpha = \log(16q/k^2)$, HOÜEL 1 (p. 57), $\theta = 0(1^g)50^g$
3D, log q, MONTESSUS DE BALLORE 1, $\theta = 0(1°)86°(30')89°(20')89°40', 89°55'$
15D, $-\log q$, NBSCL 2, $k^2 = 0(.01)1$
10D, log q, HAYASHI 1, $k^2 = 0(.001)1$
8D, log q, HAYASHI 1, $k^2 = 0(10^{-7})10^{-5}(10^{-5}).003$, $1 - k^2$ = same
7D, log q, NAGAOKA & SAKURAI 1, $k^2 = 0(.001)1$, Δ
7D, log (q/k^2) and ln (k^2/q), NAGAOKA & SAKURAI 1, $k^2 = 0(.001).05$, Δ
5D, log q, GROVER 6 (p. 251), $k^2 = 0(.005).1$, Δ^2

III-C. *log log* $(1/q)$ *and log ln* $(1/q)$

12D, log log $(1/q)$, VERHULST 1, $\theta = 0(0°.1)45°$. Computed by LOXHAY. From 0 to 15°, values to 14 decimals are given, but they are inaccurate, as in LEGENDRE's table of log K, on which they are based. See Part III
10D, log log $(1/q)$, INNES 2, $\theta = 0(1°)45°$, from VERHULST 1
7D, log log $(1/q)$, BERTRAND 1, POTIN 1, $\theta = 0(0°.5)90°$
7D, log ln $(1/q)$, NAGAOKA & SAKURAI 1, $k^2 = 0(.001)1$, Δ
5D, log ln $(1/q)$, GROVER 6 (p. 251), $k^2 = 0(.005).1$, Δ^2

Section IV. THETA FUNCTIONS OF ZERO ARGUMENT

The general notation for theta functions which we shall use is

$$\vartheta_1(x) = 2q^{\frac{1}{4}}(\sin x - q^2 \sin 3x + q^6 \sin 5x - q^{12} \sin 7x + \ldots),$$
$$\vartheta_2(x) = 2q^{\frac{1}{4}}(\cos x + q^2 \cos 3x + q^6 \cos 5x + q^{12} \cos 7x + \ldots),$$
$$\vartheta_3(x) = 1 + 2q \cos 2x + 2q^4 \cos 4x + 2q^9 \cos 6x + \ldots,$$
$$\vartheta_4(x) = 1 - 2q \cos 2x + 2q^4 \cos 4x - 2q^9 \cos 6x + \ldots,$$

where, as in Section III above, $q = e^{-\pi K'/K}$. This is the functional notation used, for example, in WHITTAKER & WATSON, and we shall adhere to it in spite of the fact that definitions in which x is replaced by πv have been adopted in important tabulations.

In this section we are concerned only with theta functions and their x-derivatives at $x = 0$. Since $\vartheta_1(0)$ vanishes, $\vartheta_1'(0)$ is considered in its place, and we have

$$\begin{aligned}\vartheta_1' &= \vartheta_1'(0) = 2q^{\frac{1}{4}}(1 - 3q^2 + 5q^6 - 7q^{12} + \ldots),\\ \vartheta_2 &= \vartheta_2(0) = 2q^{\frac{1}{4}}(1 + q^2 + q^6 + q^{12} + \ldots),\\ \vartheta_3 &= \vartheta_3(0) = 1 + 2q + 2q^4 + 2q^9 + \ldots,\\ \vartheta_4 &= \vartheta_4(0) = 1 - 2q + 2q^4 - 2q^9 + \ldots.\end{aligned}$$

On account of the difference of notation mentioned above, the $\vartheta_1'(0)$ of NAGAOKA & SAKURAI and HAYASHI is our $\pi\vartheta_1'$, and will be so described.

The four functions just defined are connected by JACOBI's identity $\vartheta_1' = \vartheta_2\vartheta_3\vartheta_4$. Moreover, each is expressible in terms of K (and moduli), as follows:

$$\vartheta_1' = \sqrt{(8kk'K^3/\pi^3)}, \qquad \vartheta_2 = \sqrt{(2kK/\pi)},$$
$$\vartheta_3 = \sqrt{(2K/\pi)} = \sqrt{(1/M)}, \qquad \vartheta_4 = \sqrt{(2k'K/\pi)}.$$

If $M = 1/\vartheta_3^2 = 1 - 4g_1q + 4g_2q^2 - 4g_3q^3 + \ldots$, the g's are all integral, and are given up to g_{27} in GLAISHER 2.

NAGAOKA & SAKURAI 1 and HAYASHI 1 also tabulate quotients which, in our notation, are

$$\pi^2\vartheta_1'''/\vartheta_1', \qquad \pi^2\vartheta_2''/\vartheta_2, \qquad \pi^2\vartheta_3''/\vartheta_3, \qquad \pi^2\vartheta_4''/\vartheta_4.$$

In their notation, the π^2 factors are absent. Mainly in consequence of JACOBI's identity $\vartheta_1' = \vartheta_2\vartheta_3\vartheta_4$, the first quotient is the sum of the other three. The expressions for ϑ_1''', ϑ_2'', ϑ_3'', ϑ_4'' are evidently

$$\begin{aligned}\vartheta_1''' &= -2q^{\frac{1}{4}}(1 - 27q^2 + 125q^6 - \ldots),\\ \vartheta_2'' &= -2q^{\frac{1}{4}}(1 + 9q^2 + 25q^6 + \ldots),\\ \vartheta_3'' &= -8(q + 4q^4 + 9q^9 + \ldots),\\ \vartheta_4'' &= 8(q - 4q^4 + 9q^9 - \ldots).\end{aligned}$$

It should not be overlooked that the quotients mentioned are expressible in terms of K and E (and moduli), as follows:

$$\pi^2\vartheta_1'''/\vartheta_1' = 4K\{(2 - k^2)K - 3E\}, \qquad \pi^2\vartheta_2''/\vartheta_2 = -4KE,$$
$$\pi^2\vartheta_3''/\vartheta_3 = -4K(E - k'^2K), \qquad \pi^2\vartheta_4''/\vartheta_4 = 4K(K - E).$$

Other notations for theta functions will be mentioned in Section X.

IV-A. ϑ_1', ϑ_2, ϑ_3, ϑ_4

15D, ϑ_2/ϑ_4[$= \mathrm{D}(90)$], ϑ_4/ϑ_2, SPENCELEY 1, $\theta = 0(1°)90°$
10–12D, ϑ_2[$= \mathrm{H}(K)$], ϑ_4[$= \Theta(0)$], ϑ_3/ϑ_4[$= C(0) = 1/\sqrt{k'}$], HIPPISLEY 1, $\theta = 0(5°)80°(1°)90°$
9D, $\log \vartheta_3$ [column $\log \Theta(\omega)$], BERTRAND 1 (p. 714), $\theta = 0(0°.5)90°$
9D, $\log \vartheta_2$ [column $\log \vartheta_2(0)$], POTIN 1 (p. 832), $\theta = 0(0°.5)90°$
8D, $\pi\vartheta_1'$ [col. $\vartheta_1'(0)$], ϑ_2, ϑ_3, ϑ_4 [col. $\vartheta_0(0)$], HAYASHI 1, $k^2 = 0(.001).5$
5D, $\pi\vartheta_1'$ [col. $\vartheta_1'(0)$], ϑ_2, ϑ_3, ϑ_4 [col. $\vartheta_0(0)$], HAYASHI 2, $k^2 = 0(.001).5$
7D, logs of these 4 functions, NAGAOKA & SAKURAI 1, $k^2 = 0(.001).5$, Δ
7D, $\log(\pi\vartheta_1'/\sqrt{k})$, NAGAOKA & SAKURAI 1, $k^2 = 0(.001).05$, Δ

Use may also be made of the values for zero argument in the small double-entry tables of HOÜEL 1 and JAHNKE & EMDE 1 (see Section X). It will be noticed that ϑ_2/ϑ_4 (see SPENCELEY 1 and HIPPISLEY 1 above) is merely an algebraic function of the modulus.

IV-B. ϑ_1''', ϑ_2'', ϑ_3'', ϑ_4''

8D, $\pi^2\vartheta_1'''/\vartheta_1'$, $\pi^2\vartheta_2''/\vartheta_2$, $\pi^2\vartheta_3''/\vartheta_3$, $\pi^2\vartheta_4''/\vartheta_4$, HAYASHI 1, $k^2 = 0(.001).5$
7D, logs of these 4 quotients, NAGAOKA & SAKURAI 1, $k^2 = 0(.001).5$, Δ
7D, $\log(-\pi^2\vartheta_3''/k^2\vartheta_3)$, $\log(\pi^2\vartheta_4''/k^2\vartheta_4)$, NAGAOKA & SAKURAI 1, $k^2 = 0(.001).05$, Δ

Section V. Laplace Coefficients and Related Functions

The Laplace coefficients $b_s^{(i)}$ occur in the expansion

$$(1 + \alpha^2 - 2\alpha \cos x)^{-s} = \tfrac{1}{2} b_s^{(0)} + \sum_{i=1}^{\infty} b_s^{(i)} \cos ix,$$

where $s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots$. The expansion is used in dynamical astronomy, where α is most often equal to the ratio of the mean distances of two planets from the Sun. The coefficients, and their derivatives with respect to α, are all linear functions of the complete elliptic integrals of the first and second kinds with modulus $k = \alpha$; in particular,

$$b_{\frac{1}{2}}^{(0)} = 4K/\pi, \qquad b_{\frac{1}{2}}^{(1)} = 4(K - E)/\pi k.$$

The principal tables multiply the Laplace coefficients by various integral powers of α and $\sqrt{(1 - \alpha^2)}$ before tabulation, or use variant forms of the functions. The argument is usually α, $\log \alpha$ or $\alpha^2/(1 - \alpha^2)$, that is, in the notation of our other sections, k, $\log k$ or $\tan^2 \theta$.

A full description of the various tables entails some algebraical complication; for details one may consult p. 333–335 of FMR *Index*. When all derivatives of all Laplace coefficients are considered, the functions may be regarded as somewhat general hypergeometric functions rather than merely as standard forms of elliptic integrals. Consequently only a few brief indications are given here.

Tables with argument α. The most extensive tables are in Runkle 1 (on errors, see Witt 2). Brown & Brouwer 1 has minor tables with argument α. See also Robbins 2 (and Innes 5), Fletcher 1, Witt 2, Norén & Raab 1, Andoyer 1, and Legendre 2 (p. 289), 5 (p. 551).

Tables with argument $\log \alpha$. Masal 1 and Gyldén 1 are extensive. The integrals tabulated in Masal 1, namely

$$\beta_n^{(s)} = (2/\pi) \int_0^{\frac{1}{2}\pi} (1 - \alpha^2 \sin^2 \phi)^{-\frac{1}{2}s} \sin^{2n} \phi \, d\phi,$$

are perhaps more germane to this *Guide* than most functions discussed in the present section.

Tables with argument $\alpha^2/(1 - \alpha^2)$. This is the argument of the major tables in the important Brown & Brouwer 1.

Coefficients used in calculating Laplace coefficients. See Runkle 1, Leverrier 1, Newcomb 1, Wellmann 1, Brown 1, 2, Williams 1, Witt 2, Brown & Brouwer 1, and Brown & Shook 1.

Section VI. Combinations of K and E occurring in connection with Currents in Circles and Coils

Below are listed a number of standard tables, many of which may be regarded as source tables, of four main kinds designed for electrical use. Some of them have doubtless been copied into various electrical text-books and hand-books, but no attempt has been made to extend the lists below to include all these.

The function f_1[$= M/4\pi\sqrt{(Aa)}$, in a common notation] occurs in connection with the mutual inductance of two circles with a common axis; f_2 in connection with the attraction between currents in such circles. The function f_3 (often called K) occurs in connection with the self-inductance of cylindrical solenoids, as Nagaoka's correction factor for finite length; in this case $\tan \theta$ is the ratio of diameter to length of the solenoid, and f_3 tends to unity for long coils (small θ). As the signs will be found to be wrong or misleading in a number of places in the literature, it may be added that f_1, f_2 and f_3 are all positive when defined as below.

Spielrein's function $f(\alpha) = f_4$ occurs in connection with the self-inductance of flat coils. If the windings lie in a plane between two concentric circles of radii a, A, and the number of turns, N, is large, each turn being approximately circular, the self-inductance in absolute electromagnetic units is $N^2 A f(\alpha)$, where $\alpha = a/A < 1$. It will be observed that $f(\alpha)$ involves integrals of K and E with respect to the modulus k, so that it is essentially

more complicated than the other three functions, but it is convenient to include it here on account of the electrical relatedness. It may be noted that

$$\int_0^1 Kdk = 2G, \qquad \int_0^1 Edk = G + \tfrac{1}{2},$$

where G is CATALAN'S constant,

$$G = \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} \phi \operatorname{cosec} \phi d\phi = 1^{-2} - 3^{-2} + 5^{-2} - 7^{-2} + \cdots = 0.91596\ 55941\ 77219,$$

so that $f(0) = \frac{3}{2}\pi(2G - 1) = 6.96957\ 04257$.

GROVER 1, 6 (p. 252) give tables relating to the maximum attraction between currents in two circles with a common axis, and the distance between the planes of the circles which produces this maximum.

Further references will be found in ROSA & GROVER 1, in GROVER 6, and in the extensive bibliography of HAK 1 (687 items, p. 232–246).

VI–A$_1$. $f_1 = (2/k - k)K - 2E/k$

7D, $\log f_1$, MAXWELL 1, $\theta = 60°(0°.1)90°$, reprinted in MASCART & JOUBERT 1, A. GRAY 1$_1$, and FOWLE 1$_1$, and (partly corrected by comparison with ROSA *et al.*) in A. GRAY 1$_2$

7D, $\log f_1$, ROSA & COHEN 1 and ROSA & GROVER 1, $\theta = 60°(0°.1)90°$, Δ, recomputation of MAXWELL 1

5D, f_1, JAHNKE & EMDE 1$_1$ (p. 77), $\theta = 60°(0°.1)90°$, antilogs of MAXWELL 1

4–6D, $4\pi f_1$, PIDDUCK 1, $\theta = 5°(6')89°(1')89°54'$

About 6S, $4\pi f_1$, log $(4\pi f_1)$, NAGAOKA & SAKURAI 2 (p. 140), $k^2 = 0(.001)1$, Δ^2

About 6S, $4\pi f_1/1000$, log $(4\pi f_1/1000)$, GROVER 5, $k^2 = 0(.005)1$, Δ^n, also auxiliary tables

4S, $2\pi k f_1$, CURTIS & SPARKS 1, $k^2 = 0(.01)1$, Δ, also auxiliary tables

4–5S, $4\pi f_1/1000$, log $(4\pi f_1/1000)$, GROVER 6 (p. 79), $k'^2 = 0(.01)1$, Δ, also auxiliary tables

About 4S, $4\pi f_1/1000$, BUREAU OF STANDARDS 1, TERMAN 1 (p. 69), $k' = .01(.001).016(.002).05(.01).95(.002)1$

8D, ϵ and log $(1 + \epsilon)$, NAGAOKA 2, $q = .02(.002).1(.001).15$, Δ for log $(1 + \epsilon)$, where $f_1 = 4\pi q^{\frac{1}{2}}(1 + \epsilon)$. Also two further auxiliary functions with q' as argument. All reproduced in ROSA & GROVER 1. Brief tables of the same kind in NAGAOKA 1. NAGAOKA 5 tabulates log $(1 + \epsilon)$ to 7D with Δ for $k = 0(.001).2$, where $k = q_1 = q^2$, q and q_1 corresponding to k and LANDEN'S modulus k_1, see Section I. Brief table of the same kind in NAGAOKA 3. (ϵ is not to be confused with the ϵ of Section III, which is denoted by $\frac{1}{2}l$ in NAGAOKA.)

VI–A$_2$. $f_2 = k\{(2 - k^2)E/(1 - k^2) - 2K\} = \sin\theta\{(1 + \sec^2\theta)E - 2K\}$

8D, log f_2, ROSA, DORSEY & MILLER 1, $\theta = 55°(0°.1)70°$, Δ^2, recomputation of RAYLEIGH & SIDGWICK 1

7D, log f_2, RAYLEIGH & SIDGWICK 1, $\theta = 55°(0°.1)69°.9$, reprinted in MASCART & JOUBERT 1$_2$

About 5S, f_2, JAHNKE & EMDE 1$_1$ (p. 79), $\theta = 55°(0°.1)69°.9$, antilogs of RAYLEIGH & SIDGWICK 1

About 6S, πf_2, log (πf_2), NAGAOKA & SAKURAI 2 (p. 161), $k^2 = 0(.001)1$, Δ^2

5D, log (πf_2), NAGAOKA 4, $q = .02(.002).1(.001).15$, Δ

About 4S, πf_2, GROVER 6 (p. 250), $k^2 = 0(.01)1$, Δ^2, abridged from NAGAOKA & SAKURAI 2

VI–A$_3$. $f_3 = (4/3\pi)[(K\cos^2\theta - E\cos 2\theta)\sin^{-2}\theta\cos^{-1}\theta - \tan\theta]$

6D, f_3, log $(\pi^2 f_3)$, NAGAOKA & SAKURAI 2, $k^2 = 0(.001)1$, $\tan\theta = 0(.001)1$, $\cot\theta = 0(.001)1$, Δ^2

6D, f_3, HAK 1 (p. 16), $\tan\theta = 0(.001).002(.002).01(.01)1$, $\cot\theta = 0(.001).002(.002).01(.01)1$

6D, f_3, log f_3, NAGAOKA 2, $\theta = 0(1°)90°$, Δ^2

6D, f_3, ROSA & GROVER 1 (T.XX), $\theta = 0(1°)90°$, Δ^2, from NAGAOKA 2

6D, f_3, NAGAOKA 2, $\tan\theta = 0(.01)1(.05)2(.1)5(.5)10$, Δ

6D, f_3, ROSA & GROVER 1 (T.XXI), $\tan\theta = 0(.01)1(.05)2(.1)5(.5)10$, Δ^2 to 5, then Δ^3, from NAGAOKA 2 with correction
6D, f_3, GROVER 6 (p. 144), $\cot\theta = 0(.01)1$, $\tan\theta = 0(.01)1$, Δ^2
5D, f_3, GROVER 3, $\tan\theta = 0(.01)1(.05)2(.1)5(.5)10(1)20$(var.)400, Δ
4D, f_3, BUREAU OF STANDARDS 1, $\tan\theta = 0(.05)2(.1)5(.2)8(.5)10(1)20$(var.)100, Δ
5D, $2\pi^2 f_3 \tan\theta$, ROSA & COHEN 1 (T.IV), ROSA & GROVER 1 (T.IV), $\tan\theta = .2(.1)1(.2)4$
4D, $2\pi^2 f_3 \tan\theta$, RUSSELL 1, $\tan\theta = .2(.1)1(.2)4$
4D, $\frac{1}{2}\pi f_3 \tan\theta$, EMDE 1, $\tan\theta = 0(.01)1$, $\cot\theta = 0(.01)1$
4–5S, $\pi^2 f_3 \tan\theta$, $\pi^2 f_3 \tan^2\theta$, JAHNKE & EMDE 1_5–1_4, BOLL 1 (p. 344), $\tan\theta = 0(.01)1$, $\cot\theta = 0(.01)1$
2–3D, $\pi^2 f_3$ (col. k), $\pi^2 f_3 \tan\theta$ (col. F), GROVER 4, 6 (p. 152), $\tan\theta = 0(.01)1$, $\cot\theta = 0(.01)1$

VI–A$_4$. $f_4 = 16\pi/[3(1-\alpha)^2]\int_\alpha^1 (1-\alpha^2 k^{-4})(K-E)dk$

6–7S, $f(\alpha) = f_4$, SPIELREIN 1, GROVER 2 (p. 570), $\alpha = 0(.05).9, 1$
4S, $f(\alpha) = f_4$, SPIELREIN 1, $\alpha = 0, .05(.01).95, .99, 1$
4S, $Q = (1-\alpha)^2 f_4$, BUTTERWORTH 1, $r_1/r_2 = \alpha = 0(.05)1$
5S, $\Phi = f_4/(1+\alpha)$, HAK 1 (p. 18), $\gamma = \alpha = 0(.05).9$
4–5S, $\Phi = f_4/(1+\alpha)$, HAK 1 (p. 18), $\rho = (1-\alpha)/(1+\alpha) = 0(.05)1$
5S, $P = 2f_4/(1+\alpha)$, GROVER 6 (p. 113), $c/2a = (1-\alpha)/(1+\alpha) = 0(.01)1$, Δ

Section VII. MISCELLANEOUS TABLES INVOLVING COMPLETE ELLIPTIC INTEGRALS AND THETA FUNCTIONS OF ZERO ARGUMENT

1. LEGENDRE functions $P_n(x)$, $Q_n(x)$ of half-odd-integral order n may be expressed in terms of complete elliptic integrals of the first and second kinds. BATEMAN[1] gives a treatment of the case $x \geqslant 1$, which arises in solving LAPLACE's equation in toroidal coordinates. For references to early work, see BATEMAN[1] and AIREY 1. Some authors (including AIREY 1 and LEVY & FORSDYKE 1) use a variant notation for toroidal functions, in that they write $Q_0, Q_1, Q_2, \cdots$ in place of $Q_{-\frac{1}{2}}, Q_{\frac{1}{2}}, Q_{\frac{3}{2}}, \cdots$, in order to avoid fractional suffixes; but we shall adhere to the usual Legendre function notation. Some formulae for low n have recently been stated explicitly by MILLER[2], for both cases, $-1 \leqslant x \leqslant 1$ and $x \geqslant 1$.

When $|x| \leqslant 1$, put $x = \cos\theta$. Then the elliptic modulus being given by

$$k^2 = \tfrac{1}{2}(1-x) = \sin^2\tfrac{1}{2}\theta, \qquad k'^2 = \tfrac{1}{2}(1+x) = \cos^2\tfrac{1}{2}\theta,$$

we have, for instance,

$$P_{-\frac{1}{2}}(x) = 2K/\pi, \qquad P_{\frac{1}{2}}(x) = 2(2E-K)/\pi,$$

and (see MILLER[2])

$$Q_{-\frac{1}{2}}(x) = K' \mp iK, \qquad Q_{\frac{1}{2}}(x) = 2E' - K' \pm i(2E-K).$$

Further P's and Q's may be obtained from the usual recurrence relation. No tables, except of course for $P_{-\frac{1}{2}}$ and $Q_{-\frac{1}{2}}$, are known, but they could evidently be very easily formed, with argument θ, from the tables of LEGENDRE or HEUMAN 1.

When $x \geqslant 1$, put $x = \cosh\sigma$. Then the elliptic modulus being given by

$$k^2 = 2/(x+1) = \operatorname{sech}^2\tfrac{1}{2}\sigma, \qquad k'^2 = (x-1)/(x+1) = \tanh^2\tfrac{1}{2}\sigma,$$

we have, for instance,

$$P_{-\frac{1}{2}}(x) = 2kK'/\pi, \qquad P_{\frac{1}{2}}(x) = 2(2E' - k^2K')/k\pi,$$
$$Q_{-\frac{1}{2}}(x) = kK, \qquad Q_{\frac{1}{2}}(x) = [(2-k^2)K - 2E]/k.$$

The values of $Q_n(x)$, $Q_n'(x)$, $Q_n''(x)$ are given to 6D or 7S for $n = -\frac{1}{2}(1)+2\frac{1}{2}$, $\sin^{-1}k = 83°, 80°(5°)65°(10°)25°$ in LEVY & FORSDYKE 1. The values of $P_n(x)$, $P_n'(x)$, $Q_n(x)$, $Q_n'(x)$, as well as corresponding associated Legendre functions and their first derivatives, are given to about 6S for $n = -\frac{1}{2}(1)+4\frac{1}{2}$, $x = 1(.1)10$ in NBSCL 1. $P_n(\cosh\sigma)$, $P_n'(\cosh\sigma)$, $Q_n(\cosh\sigma)$, $Q_n'(\cosh\sigma)$ are tabulated to 4D for $n = -\frac{1}{2}(1)+\frac{1}{2}$, $\sigma = 0(.1)3$ in FOUQUET 1.

2. F. W. NEWMAN 1 has a number of tables in which the argument is $\rho = \pi K'/2K$, so that $q = e^{-2\rho}$, $\rho = \frac{1}{2} \ln (1/q)$. For the first of these tables, see Section XI. The remainder are as follows:

p. 129, 16D, $\frac{1}{2} \ln (1/k')$, $\rho = 1(.1)6.3$
p. 130, 16D, $\frac{1}{2} \ln (2K/\pi) = \ln \vartheta_3$, $\rho = 1(.1)6.3$
p. 131, 16D, $\frac{1}{2}(2K/\pi - 1)$, $\rho = 1(.1)6.1$
p. 132, 16D, $\ln Q$, $\rho = 1(.1)4.6$, where $Q^{-1} = (1 - q^2)(1 - q^4)(1 - q^6) \cdots = (\vartheta_1'/2q^{\frac{1}{4}})^{\frac{1}{3}}$
p. 133, 16D, $\frac{1}{2}(1 - 2k'K/\pi)$, $\rho = 1(.1)6.3$
p. 134, 16D, kK/π, $\rho = 1(.1)6$
p. 135, 18D, $e^{-\rho}$, $\rho = .1(.1)37$

The last table barely falls within the scope of this *Guide*, but is included to complete the description of NEWMAN's ρ-tables, and is relevant to the remainder of these. It is taken from NEWMAN's original publication.[3] About 16D are correct.

3. It is well known that, on putting $q = e^{-\pi t}$, each term of the defining expansions of the four theta functions satisfies $\partial^2 f/\partial x^2 = \partial f/\partial t$, the diffusion or heat conductivity equation. The theta functions themselves are not usually the appropriate combinations of such terms in physical problems. But integrals of theta functions of zero argument have been tabulated in connection with a number of diffusion problems. For example,

$$\frac{4}{\pi^2} \int_0^t \vartheta_2 dt = 1 - \frac{8}{\pi^2} \left(e^{-t} + \frac{1}{9} e^{-9t} + \frac{1}{25} e^{-25t} + \cdots \right),$$

and the expression on the right is tabulated as the "simple diffusion function" in MCKAY 1 to 4D for $t = 0(.01)3.59$. Similarly a table in OLSON & SCHULTZ 1 involves $\int_0^t \vartheta_1' dt$, some in SHERWOOD 1 involve $\int_0^t \vartheta_2 dt$ and $\int_0^t (\int_0^t \vartheta_2 dt) dt$, some in A. B. NEWMAN 1 involve first integrals (with respect to t) of ϑ_2 and ϑ_3, some in A. B. NEWMAN 2 involve first and second integrals of ϑ_2 and ϑ_3, and some in A. B. NEWMAN 3 involve first and second integrals of ϑ_3 and ϑ_4. These tables have various arguments (t/π^2, $4t/\pi^2$, $4t$), and particulars will be omitted here (some will be found in FMR *Index*, p. 323). There appears to be need for a consolidated table of first and second integrals of theta functions of zero argument with respect to t or some simple multiple of t. It may be noted that F. W. NEWMAN'S ρ is $2t$.

4. EMDE 2 proposes new normal forms of complete elliptic integrals, namely, quantities D, B, C defined by

$$D = (K - E)/k^2,$$
$$B = K - D = (E - k'^2 K)/k^2,$$
$$C = (D - B)/k^2 = [(2 - k^2)K - 2E]/k^4.$$

In the notation of Section VI, $C = f_1/k^2$. EMDE 2 tabulates D, B, C to 4–5D with $I\Delta$ for $k^2 = 0(.01)1$. For D and C the auxiliary functions

$$d = D + 1 - \ln (4/k'),$$
$$c = C + 2 - \ln (4/k')$$

are tabulated to 3–4D with $I\Delta$ for $k^2 = .7(.01)1$. These five tables are reproduced in JAHNKE & EMDE 1_3, 1_4. CECCONI 1 gives $(E - k'^2 K)/k^2$, which he calls G, to 4D for $k = 0(.01)1$; see also Section II–A_3.

5. HILL 1 tabulates to 8D with Δ^2 for $\theta = 0(0°.1)50°$ the logarithms of three quantities $\mathfrak{R}$, $\mathfrak{Q}'$ and $\mathfrak{N}$ which may be identified as follows:

$$\pi k'^2 \mathfrak{R} = 2E,$$
$$k^4 k'^2 E \mathfrak{Q}' = 2(1 - k^2 k'^2)E - k'^2(1 + k'^2)K,$$
$$k^2 k'^2 E \mathfrak{N} = (1 + k^2)E - k'^2 K.$$

6. SMEKAL 1 has a table of quantities K, $Q(K)$, $Q(1/K)$, $1/K$ with argument α. This notation is so inconvenient for present purposes that the tabulated functions will be denoted instead by k, Q, Q^*, $1/k$. The primary argument of the table is

$$\alpha = 45°(3')44°30'(6')40°(15')35°(30')5°.$$

k is tabulated to 5D, the other functions to about 4D, all but $1/k$ having first differences. The definitions in terms of α are

$$k = \tan^2 \alpha, \qquad Q = \pi^{-1} \sin^2 \alpha (K - E \sec 2\alpha), \qquad Q^* = \pi^{-1} \cos^2 \alpha (K + E \sec 2\alpha),$$

where K and E have modular angle 2α. But k is the reduced modulus derived from $\sin 2\alpha$ by Landen's transformation (Section I), and in terms of this subsidiary argument we have

$$Q = -2k^2 dK/(\pi dk) = -2k(E - k'^2 K)/(\pi k'^2),$$
$$Q^* = 2d(kK)/(\pi dk) = 2E/(\pi k'^2),$$

where K and E now have modulus k. Thus Q is related to EMDE's B and Q^* is identical with HILL's $\mathfrak{Q}$ (see 4 and 5 above). Replacing k now by the symbol α usual in planetary theory (not SMEKAL's α), the identifications are

$$Q = -\tfrac{1}{2}\alpha^2 b_0', \qquad Q^* = \tfrac{1}{2}(b_0 + \alpha b_0'),$$

where $\frac{1}{2}b_0$ is the constant term in the Fourier expansion of $(1 + \alpha^2 - 2\alpha \cos x)^{-\frac{1}{2}}$, and $b_0' = db_0/d\alpha$. The symbol α just introduced in place of SMEKAL's K is appropriate, because the latter is the ratio of the radii of two circular orbits (compare Section V).

7. Coefficients in the Fourier expansions of two Jacobian functions (see Sections X and XI) are tabulated in GLAISHER 1. If

$$\left.\begin{aligned} 2 \operatorname{sn} u \operatorname{cn} u &= \Sigma A_n \sin 2nx \\ 2 \operatorname{am} u - 2x &= \Sigma B_n \sin 2nx \end{aligned}\right\} x = \pi u/2K,$$

we have

$$A_n = n \left(\frac{\pi}{kK}\right)^2 \frac{4q^n}{1 + q^{2n}}, \qquad B_n = \frac{1}{n} \frac{4q^n}{1 + q^{2n}}.$$

GLAISHER 1 gives A_1, B_1, about 7D, $\theta = 0(1°)89°$; $\log A_1$, $\log B_1$, about 8D, $\theta = 0(1°)89°$; A_2, B_2, A_3, B_3, 7D, $\theta = 0(5°)85°$.

8. SHOOK 1 tabulates the arithmetic-geometric mean of 1 and x to 4S for $x = 1(.1)20(1)100$.

9. DEVISON 1 tabulates (p. 334) $\frac{1}{2}\gamma(t) = \tan^{-1}(K'/K)$ to 3D of a radian for $t = k^2 = 0(.00001).0001(.0001).0005(.0002).0015, .002(.001).010, .013, .016, .020(.005).05(.01).10(.02).2(.1).5$.

This small table is then used for the evaluation by numerical quadrature of integrals such as $\int_0^1 \frac{\gamma(t)dt}{t - \lambda}$. See *MTAC*, v. 2, p. 268, 1947.

10. See also GREAT BRITAIN, ADMIRALTY COMPUTING SERVICE 1.

[1] H. BATEMAN, *Partial Differential Equations of Mathematical Physics*, Cambridge, 1932 and New York, 1944, p. 461.
[2] J. C. P. MILLER, *Math. Gazette*, v. 30, 1946, p. 240.
[3] F. W. NEWMAN, Camb. Phil. Soc., *Trans.*, v. 13, pt. 3, 1883, p. 145–241.

Section VIII. INCOMPLETE ELLIPTIC INTEGRALS OF THE FIRST AND SECOND KINDS

Almost all tables of $F(\phi)$ and $E(\phi)$ are derived ultimately from the great Table IX of LEGENDRE (it is to be noted that SAMOĬLOVA-ĨAKHONTOVA states that her tables were constructed by interpolation in LEGENDRE). The chief exception is the independently-computed table of $E(\phi)$ by SCHMIDT, which gives more values to more decimals than LEGENDRE's table of $E(\phi)$, but has the unorthodox arguments k' and $\sin \phi$; the values are arcs of ellipses with semi-axes $a = 1$, $b = k'$ between the points $(0, k')$ and $(\sin \phi, k' \cos \phi)$. HOÜEL's second table under **B** below is also probably independent of LEGENDRE. No attempt has been made to seek out all abbreviations of LEGENDRE's table, which is now easily accessible through the three facsimile reproductions of POTIN, EMDE and PEARSON.

For the second integral with argument of Jacobian type, see Section X.

KAPLAN 2 has recently given important auxiliary tables for the computation of $F(\phi)$ and $E(\phi)$ when both θ and ϕ are near 90°, so that interpolation in LEGENDRE is difficult or impossible. KAPLAN puts

$$r = k'/k = \cot\theta, \qquad x = \cos\phi,$$
$$K - F(\phi) = (2/\pi)K' \sinh^{-1}(x/r) + x(r^2 + x^2)^{\frac{1}{2}}f,$$
$$E - E(\phi) = (2/\pi)(K' - E') \sinh^{-1}(x/r) + x(r^2 + x^2)^{\frac{1}{2}}e,$$

and tabulates the auxiliary functions f and e to 10D for $x^2 = -.005(.005) + .160$ and $r^2 = -.005(.005) + .160$, the negative arguments $-.005$ being included to facilitate interpolation.

As an example of one of the more extended applications of the incomplete integrals to the formation of tables of physical interest, see OSBORN 1.

VIII-A. *$F(\phi)$ and $E(\phi)$*

12D, LEGENDRE 3 (T.II), 5 (T.II), $\theta = 45°$, $\phi = 0(0°.5)90°$, Δ^4 or Δ^5
12D, LEGENDRE 3 (T.VIII), 5 (T.VIII), 7, $\theta = 0(1°)90°$, $\phi = 45°$, Δ^3 or Δ^4
10D, LEGENDRE 3 (p. 84), 5 (p. 77), $\theta = 89°$, $\phi = 0(0°.5)90°$, Δ
9–10D, LEGENDRE 3 (T.IX), 5 (T.IX), 6, 7, 8, $\theta = 0(1°)90°$, $\phi = 0(1°)90°$
10D, KAPLAN 2, $\theta = 70°(5°)85°$, $\phi = 70°(5°)85°$
5D, LÉVY 1, BOHLIN 1, DALE 1, HANCOCK 1, HAYASHI 2, $\theta = 0(5°)90°$, $\phi = 0(1°)90°$
5D, $F(\phi)$, FRICKE 1, $\theta = 0(5°)90°$, $\phi = 0(1°)90°$
5D, $E(\phi)$, FRICKE 1, $\theta = 0(10°)90°$, $\phi = 0(2°)90°$
5D, BOLL 1 (p. 352), $\theta = 0(3°)90°$, $\phi = 0(3°)90°$
5D, KIEPERT 1, $\theta = 0(5°)90°$, $\phi = 0(5°)90°$
5D, BERTRAND 1, $\theta = 0(15°)90°$, also 10°, 80°, 89°, $\phi = 0$(var.)$65°(1°)90°$
5D, LÁSKA 1, $\theta = 0(15°)90°$, also 10°, 80°, $\phi = 0(1°)5°(5°)85°(1°)90°$
5D, $F(\phi)$ only, GREENHILL 3, $\theta = 45°$, $\phi = 0(0°.5)90°$
4D, GLAZENAP 1, $\theta = 0(5°)90°$, $\phi = 0(1°)90°$; also $F(\phi)$, $\theta = 87°, 89°$, $\phi = 0(1°)90°$
4D, JAHNKE & EMDE 1, $\theta = 0(5°)90°$, $\phi = 0(1°)90°$, also $F(\phi)$, $\theta = 89°$, $\phi = 0$(var.) $65°(1°)90°$; $F(\phi)$ has 5D for $\phi = 0(1°)5°$
4D, SILBERSTEIN 1, $\theta = 0(5°)90°$, $\phi = 0(1°)90°$; $F(\phi)$ has 5D for $\phi = 0(1°)5°$
4D, ROSENBACH, WHITMAN & MOSKOVITZ 1, $\theta = 5°(5°)90°$, $\phi = 0(1°)90°$
4D, PEIRCE 1_5–1_8, $\theta = 0(15°)90°$, also 10°, 80°, $\phi = 0(1°)5°(5°)85°(1°)90°$
4D, LIGOWSKI 1, HÜTTE 1, $\theta = 0(10°)90°$, $\phi = 0(10°)90°$
3D, MONTESSUS DE BALLORE 1, $\theta = 0(15°)90°$, $\phi = 0(5°)90°$
3D, BYERLY 1, CAMPBELL 1, $\theta = 0(6°)30°, 37°, 45°, 53°, 64°, 90°$, $\phi = 0(5°)90°$
4D, HOÜEL 1, $\theta = 0(10^g)100^g$, $\phi = 0(10^g)100^g$
5D, SAMOĬLOVA-ĨAKHONTOVA 1, $k^2 = 0(.01)1$, $\phi = 0(1°)90°$, $I\Delta$ in k^2, Δ in ϕ
4D, $F(\phi)$ only, RENO 1, $k^2 = 0(.01)1$, $\phi = 0(1°)90°$
3D, GRIFFIN 1, $k = 0(.1).9, .95, .99, 1$, $\phi = 0(10°)90°$
11D, $E(\phi)$ only, SCHMIDT 1, $k' = 0(.01)1$, $\sin\phi = 0(.01)1$, Δ^2; also 10D, $\sin\phi = .990(.001).999$, $k' = .80(.01).99$, Δ^5 in k'

VIII-B. *Log $F(\phi)$ and log $E(\phi)$*

4D, HOÜEL 1, $\theta = 0(10^g)100^g$, $\phi = 0(10^g)100^g$
4D, log $F(\phi)$ only, HOÜEL 1, $\theta = 90^g(1^g)100^g$, $\phi = 90^g(1^g)100^g$

VIII-C. *$F(\phi)/K$*

4D, HOÜEL 1, $\theta = 0(10^g)100^g$, $\phi = 0(10^g)100^g$

VIII-D. *Other combinations*

6D, $[K - F(\phi)]/\sqrt{2}$, MACGREGOR 1, $\theta = 45°$ (lemniscate case), $\phi = 0(0°.5)90°$
3D, $\int_x^\infty (1 + y^4)^{-\frac{1}{2}}dy$, BORN 1 (p. 432), $x = 0(.1)1$, $1/x = 0(.1)1$; the integral equals $K - \frac{1}{2}F(2\tan^{-1}x)$ with $\theta = 45°$

2D, $P(\alpha) = \int_0^{\alpha} \sin^{\frac{1}{2}} t\,dt = 2\int_{\frac{1}{2}\pi-\frac{1}{2}\alpha}^{\frac{1}{2}\pi} (1 - 2\sin^2 u)^{\frac{1}{2}} du$, FREEMAN 1, $\alpha = 0(5°)180°$. Also functions involving $P(\alpha)$. To reduce integrand to standard form with $k^2 = \frac{1}{2}$, put $\sin u = 2^{-\frac{1}{2}} \sin\phi$.

Section IX. Elliptic Integrals of the Third Kind

The incomplete elliptic integral of the third kind, say

$$\int_0^{\phi} \frac{1}{1 - p\sin^2\phi} \frac{d\phi}{\sqrt{(1 - k^2\sin^2\phi)}},$$

is apparently a function of three variables, k, p and ϕ.

But in the "hyperbolic" cases, in which $0 < p < k^2$ or $p > 1$, the integral may be expressed in terms of theta and Jacobian zeta functions of real arguments, so that it can be evaluated by means of double-entry tables.

In the "circular" cases, which tend to be more important in practice (the spherical pendulum provides an elementary dynamical example), we have $k^2 < p < 1$ or $p < 0$, and the arguments of the theta and zeta functions become complex. The circular integral has apparently not been tabulated as a function of three variables.

But the complete integral with upper limit of integration $\frac{1}{2}\pi$ can in all cases be expressed in terms of complete and incomplete integrals of the first and second kinds. In the circular case $k^2 < p < 1$ (to which the case $p < 0$ may be reduced), the complete integral has been tabulated in HEUMAN 1. The actual function tabulated is

$$\Lambda_0(\alpha, \beta) = \frac{2}{\pi}\int_0^{\pi/2} \frac{\cos^2\alpha \sin\beta\cos\beta\sqrt{(1 - \cos^2\alpha\sin^2\beta)}}{\cos^2\alpha\cos^2\beta + \sin^2\alpha\cos^2\phi} \frac{d\phi}{\sqrt{(1 - \sin^2\alpha\sin^2\phi)}},$$

so that α is the modular angle and $p = \sin^2\alpha/(\sin^2\alpha + \cos^2\alpha\cos^2\beta)$. The expression in terms of first and second integrals is

$$\tfrac{1}{2}\pi\Lambda_0(\alpha, \beta) = KE'(\beta) - (K - E)F'(\beta),$$

where K and E have modulus $\sin\alpha$, and primes are used to denote that $F'(\beta)$ and $E'(\beta)$ have modulus $\cos\alpha$. HEUMAN 1 gives $\Lambda_0(\alpha, \beta)$ to 6 decimals without differences for $\alpha = 0(1°)90°$, $\beta = 0(1°)90°$, with a supplementary 6-decimal table for $\alpha = 0(0°.1)5°.9$, $\beta = 80°(1°)89°$.

I know of no other table of a standard form of third elliptic integral. A rather special inversion of HEUMAN'S table is given in connection with the quantum mechanics of a freely rotating rigid body in G. W. KING 1, 2, 3; for a detailed account, see RMT 467, 544.

BARTKY 1 discusses the numerical calculation of a general form of complete elliptic integral by use of arithmetic-geometric means. Defining these by

$$m_i = \tfrac{1}{2}(m_{i-1} + n_{i-1}), \qquad n_i = (m_{i-1}n_{i-1})^{\frac{1}{2}}, \qquad m_0 = m, \qquad n_0 = n,$$

he shows that, if the difference between m_3 and n_3 may be neglected, that is, if m_3 gives a sufficiently accurate approximation to the mean $M = m_\infty = n_\infty$, then

$$\int_0^{\pi/2} \frac{F(R)d\phi}{R} = \frac{\pi}{8m_3}\{\tfrac{1}{2}F(m) + \tfrac{1}{2}F(n) + F(n_1) + F(m') + F(n')\},$$

where

$$R^2 = m^2\cos^2\phi + n^2\sin^2\phi$$

and

$$m' = n_2 + (n_2^2 - n_1^2)^{\frac{1}{2}}, \qquad n' = n_2 - (n_2^2 - n_1^2)^{\frac{1}{2}}.$$

Taking $m = 1$, $n = k'$, which gives R the Legendre normal form $(1 - k^2\sin^2\phi)^{\frac{1}{2}}$, he tabulates m_3, n_1, m', n' to $4D$ for $k' = .10(.01).49$ and to 5D for $k' = .50(.01)1$. As a special case, to calculate

$$\int_0^{\pi/2} \frac{1}{1 - p\sin^2\phi} \frac{d\phi}{(1 - k^2\sin^2\phi)^{\frac{1}{2}}},$$

take

$$F(R) = k^2(k^2 - p + pR^2)^{-1}.$$

NYSTRÖM 1 discusses the practical calculation, by graphical and planimetric methods, of the normal form

$$\Pi(k, \lambda, \phi) = \int_0^\phi \frac{\sqrt{(1 - k^2 \sin^2 \psi)}}{1 + \lambda^2 \sin^2 \psi} d\psi = \int_0^\phi \frac{\sqrt{(\cos^2 \psi + k'^2 \sin^2 \psi)}}{\cos^2 \psi + \lambda'^2 \sin^2 \psi} d\psi,$$

where $\lambda'^2 = \lambda^2 + 1 > 0$. This is a circular case. NYSTRÖM 1 gives no elliptic table, though he provides an approximate nomogram relating k', λ' and the complete integral $\Pi(k, \lambda, \frac{1}{2}\pi)$ in the ranges $0 \leqslant k' \leqslant 1$, $0 \leqslant \lambda' \leqslant 10$. But he gives double-entry tables and diagrams of the elementary functions

$$x(\lambda, \psi) = 3.6\pi^{-1} \tan^{-1} (\lambda' \tan \psi),$$
$$y(k, \psi) = \sqrt{(\cos^2 \psi + k'^2 \sin^2 \psi)},$$
$$z(k, \psi) = k' + (1 - k') \cos \psi - y(k, \psi),$$

the tables being for $\lambda' = 0(.1)1(1)10$, $k' = 0(.1).9$, $\psi = 0(5°)90°$. These enable one to evaluate $\Pi(k, \lambda, \phi)$, since it is proportional to the Stieltjes integral

$$\int_0^\phi y(k, \psi) dx(\lambda, \psi).$$

Use of the auxiliary function z in place of y gives increased accuracy, since z is in general much smaller than y, and the remaining contribution to Π is an elementary integral.

TALLQVIST 1, 2 tabulate respectively magnetic potential (solid angle) due to a circular current, and gravitational potential due to a uniform circular disc. Both may be expressed either as infinite series involving Legendre functions, or in terms of elliptic integrals, including those of the third kind.

A very complicated integral expressible in terms of the third elliptic integral is tabulated for seismological purposes in CONFORTO & VIOLA 1.

Section X. THETA FUNCTIONS, JACOBIAN AMPLITUDE, SECOND INTEGRAL AND ZETA FUNCTION

We shall use the four theta functions already defined in Section IV, in which tables of theta functions of zero argument have been listed. The connection with Jacobi's earlier notation is, if $M = \pi/2K$, and $x = Mu = \pi u/2K$,

$$\vartheta_1(x) = \vartheta_1(Mu) = \mathrm{H}(u), \qquad \vartheta_2(x) = \vartheta_1(\tfrac{1}{2}\pi - x) = \mathrm{H}(K - u),$$
$$\vartheta_4(x) = \vartheta_4(Mu) = \Theta(u), \qquad \vartheta_3(x) = \vartheta_4(\tfrac{1}{2}\pi - x) = \Theta(K - u).$$

All tables of theta functions employ, substantially, the argument x; that is, they divide the quadrant (for x) or the quarter-period K (for u) into a number (say 90 or 100) of equal parts. But MILNE-THOMSON's tables of Jacobian elliptic and zeta functions have argument u at an interval (.01) which is incommensurable with K for the tabular moduli.

The few tables of the amplitude $\phi = \mathrm{am}\, u$, defined by $u = F(\phi)$, also have x or u/K as argument.

To avoid confusion, we follow GUDERMANN and other writers in defining

$$\mathrm{el}\, u = \int_0^u \mathrm{dn}^2 u du = \int_0^\phi \sqrt{(1 - k^2 \sin^2 \phi)} d\phi = E(\phi),$$

el u being denoted more often by $E(u)$ in the literature. A quantity often tabulated instead of el u is the Jacobian zeta function, or periodic part of el u, defined by

$$\mathrm{Z}(u) = \mathrm{zn}\, u = \mathrm{el}\, u - Eu/K.$$

The GLAISHER and GREENHILL notations will be explained in connection with the descriptions of the corresponding tables.

HOÜEL 1 tabulates to 4D the common logarithms of $\vartheta_n(x)$ and $\vartheta_n'(x)/\vartheta_n(x)$, $n = 1(1)4$, for $\theta = 0(10')90°$, $x = 0(10')100°$. Natural values deduced (with correction) from these logarithms are given to 4S in JAHNKE & EMDE 1. (In both tables, our ϑ_4 is denoted simply by ϑ.)

ROSENHEAD 1 tabulates, for $\theta = 9°$, $n = 1(1)4$,

$$\vartheta_n(x), \quad \pi d\vartheta_n(x)/dx, \quad \text{to 3–4D for } x/\pi = 0(.05).5,$$
$$\vartheta_n(iy), \quad \pi d\vartheta_n(iy)/d(iy), \quad \text{to 4D for } y/\pi = 0(.05)1.$$

The GLAISHER manuscript (GLAISHER 3, now in the possession of the BAASMTC) relates to the theta functions

$$\Theta_1(x) = \vartheta_3\vartheta_1(x)/\vartheta_2 = \vartheta_1(x)/\sqrt{k}, \qquad \Theta_2(x) = \vartheta_4\vartheta_3(x)/\vartheta_2 = \vartheta_2(x)\sqrt{(k'/k)},$$
$$\Theta_3(x) = \vartheta_4\vartheta_3(x)/\vartheta_2 = \vartheta_3(x)\sqrt{k'}, \qquad \Theta_4(x) = \vartheta_4(x).$$

The four functions were computed in 1872–75 for the BAAS under GLAISHER's superintendence. They were tabulated to 8D, both naturally and logarithmically, for $\theta = 0(1°)89°$, $x = 0(1°)90°$. The tables were completely set up in type (356 pages), but were not published, and no printed copy appears to have survived. See BAAS *Reports* of 1873 (p. 171) and 1930 (p. 250), *Mess. Math.*, v. 6, p. 111–112, 1876, and *MTAC*, v. 3, p. 92, 1948.

In 1911 GREENHILL brought forward (BAASMTC 1) a scheme for the rearrangement of the elliptic tables. He suggested that tables be made of four theta functions defined by

$$A(r) = \vartheta_1(x)/\vartheta_2, \qquad B(r) = \vartheta_2(x)/\vartheta_3, \qquad C(r) = \vartheta_3(x)/\vartheta_4, \qquad D(r) = \vartheta_4(x)/\vartheta_4,$$

where $x = r°$. These have the property, which both the ordinary and the GLAISHER forms lack, that the division-values arising at bisection, trisection, etc. of the quadrant or quarter-period are algebraic functions of the modulus; values for $x = 0(3°)90°$ may, as in the case of the trigonometric functions, be expressed by surd formulae. They also have the advantage over GLAISHER's forms of retaining simple relationships between functions of complementary arguments, since

$$B(r) = A(90 - r), \qquad C(r) = D(90 - r).$$

GREENHILL suggested a semi-quadrantal tabular arrangement, giving, at one opening for each modulus,

$$u = rK/90 = F(\phi), \qquad \phi = \operatorname{am} u, \qquad E(r) = \operatorname{zn} u, \qquad D(r), \qquad A(r)$$

for $r = 0(1)90$. (The notation $E(r)$ for the zeta function $Z(u) = \operatorname{zn} u$ is unfortunate.) The experimental tables of this kind for $\theta = 15°, 45°, 75°$ in BAASMTC 2, 4 are practically superseded by the corresponding definitive tables in HIPPISLEY 1 or SPENCELEY 1, described below. Only in the improved table, calculated by HIPPISLEY, for $\theta = 45°$ (BAASMTC 4) is appreciably more information given (ϕ to 0°.001 instead of to 1′) than in HIPPISLEY's Smithsonian tables. There are, however, tables of the same kind in BAASMTC 2, 3, 4 for the "singular" moduli for which

$$K/K' = 1/\sqrt{2},\ \sqrt{2},\ 2,\ 3/\sqrt{2},\ 2\sqrt{2},\ 3,\ 2\sqrt{3},\ 4,\ 3\sqrt{2},\ 5,\ 3\sqrt{3}.$$

Details of the transformations employed are given. These tables are rather rough, but not without interest, since some of the modular angles exceed 89°, and such cases contribute appreciably to three noteworthy graphs of $D(r)$, $A(r)$ and $E(r)$ in BAASMTC 3. BAASMTC 4 also gives u, $\operatorname{zn} u$, $D(r)$, $A(r)$ to 15D for the 7 and 17 sections in the lemniscate case ($\theta = 45°$), the values of u/K or $r/90$ being $0(1/7)1$ and $0(1/17)1$ respectively. A few numerical values relating to lemniscate sections are contained in WILTON 1.

Various lemniscate constants are given in GAUSS 1_1 (p. 31), 1_2 (p. 150), 2 (p. 364, 413–421), and various rational powers of $e^{-\pi}$ to many figures in GAUSS 2 (p. 418–432) and BAASMTC 4.

The final result of the GREENHILL-HIPPISLEY work for integral degrees of modular angle was published in the Smithsonian tables of HIPPISLEY 1. These give, in columns headed

$$F(\phi),\ \phi,\ E(r),\ D(r),\ A(r),$$

the quantities $u = rK/90$, ϕ, $\operatorname{zn} u$, $D(r)$, $A(r)$ for $\theta = 5°(5°)80°(1°)89°$, $r = 0(1)90$; all values are to 10D, except those of ϕ, which are to the nearest minute. The heading for each modular angle provides values of K, K', E, E', q, $\Theta(0) = \vartheta_4$, $H(K) = \vartheta_2$; these have been listed in Sections II, III, IV.

The important GREENHILL-HIPPISLEY work has been greatly extended in the recent monumental tables of the SPENCELEYS, the most important elliptic tables ever published. These give (i) tables for each degree of θ, compared with $\theta = 5°(5°)80°(1°)89°$ in HIPPISLEY 1, (ii) Jacobian functions sn, cn, dn (see Section XI), omitted in HIPPISLEY 1, as well as theta functions, etc. as described in the next paragraph, (iii) more decimals than in HIPPISLEY 1.

Apart from three columns giving sn u, cn u, dn u, the SPENCELEY 1 tables give, in columns headed

$$u = (r/90)K = F(\phi, k),\ \phi,\ E(\phi, k),\ A(r),\ D(r),$$

values of these quantities to 12D for $\theta = 1°(1°)89°$, $r = 0(1)90$. The tabulated functions are as in the HIPPISLEY 1 tables, except that ϕ is expressed in radians and that $E(\phi, k)$ is the second integral, or el u, while HIPPISLEY's $E(r)$ is the zeta function, or zn u. The table of ϕ is by far the most extensive in existence, and will make it comparatively easy to check LEGENDRE's great double-entry tables, by inverse interpolation in SPENCELEY 1. The arrangement of all tables is quadrantal. The headings for each modular angle give values of K, K', E, E', q, q', $D(90) = \vartheta_3/\vartheta_4$, $1/D(90) = \vartheta_4/\vartheta_3$; these have been listed in Sections II, III, IV.

On a few theta functions tabulated in NBSCL 2, see *MTAC*, v. 1, p. 126, 1943.

LEGENDRE 3 (p. 96), 5 (p. 88) gives ϕ to 6D of 1″ with Δ^3 and el $u = E(\phi)$ to 11D with Δ^4 for $\theta = 45°$, $200\ u/K = 0(1)20$. LEGENDRE 3 (T.VII), 5 (T.VII) gives ϕ to 7D of 1″ with Δ^3 for $\theta = 0(0°.1)45°$, $u = K/10$.

MILNE-THOMSON 4 gives $Z(u) = \text{zn}\ u$ to 7D, with second differences in u, for $k^2 = .1(.1)1$ and for u at interval .01 from 0 to a limit 2, 2.5 or 3, depending on k^2, but always greater than K (except of course at $k^2 = 1$, where K is infinite).

FOX & MCNAMEE 1 have recently tabulated the Jacobian zeta function of complex argument. They write

$$f_1(\psi_1, \phi, \alpha) + if_2(\psi_1, \phi, \alpha) = \text{zn}\ (K\psi_1 + iK'\phi, k) + i\pi\phi/(2K),$$

the notation ψ_1 and ϕ for the fractions of the quarter-periods arising from their potential problem, and α denoting the modular angle $\sin^{-1} k$. It is unnecessary to tabulate f_1 and f_2 separately, as the authors show that $f_2(\psi_1, \phi, \alpha) = f_1(1 - \phi, 1 - \psi_1, \frac{1}{2}\pi - \alpha)$. They tabulate f_1 to 3S for $\alpha = 1°(1°)5°(5°)85°(1°)89°$, $\psi_1 = 0(.1)1$, $\phi = 0(.1)1$.

On coefficients in an expansion for am u, see Section VII–7 (GLAISHER 1).

Section XI. JACOBIAN ELLIPTIC FUNCTIONS

If $x = Mu = \pi u/2K$, we have

$$\text{sn}\ u = \frac{1}{\sqrt{k}}\frac{\vartheta_1(x)}{\vartheta_4(x)}, \qquad \text{cn}\ u = \sqrt{\frac{k'}{k}}\frac{\vartheta_2(x)}{\vartheta_4(x)}, \qquad \text{dn}\ u = \sqrt{k'}\frac{\vartheta_3(x)}{\vartheta_4(x)}.$$

The connection with the theta functions of GLAISHER 3 is

$$\text{sn}\ u = \Theta_1(x)/\Theta_4(x), \qquad \text{cn}\ u = \Theta_2(x)/\Theta_4(x), \qquad \text{dn}\ u = \Theta_3(x)/\Theta_4(x).$$

Elliptic functions may be computed from HIPPISLEY's tables by the formulae

$$\text{sn}\ u = \frac{1}{\sqrt{k'}}\frac{A(r)}{D(r)}, \qquad \text{cn}\ u = \frac{B(r)}{D(r)}, \qquad \text{dn}\ u = \sqrt{k'}\frac{C(r)}{D(r)},$$

where $r = 90u/K$.

The great SPENCELEY 1 tables give sn u, cn u, dn u to 12D for $\theta = 1°(1°)89°$, $r = 90u/K = 0(1)90$, that is, $x = 0(1°)90°$.

The NBSCL 2 manuscript gives sn u, cn u, dn u to 15D for $k^2 = 0(.01)1$, $u/K = .01, .1(.1)1$, that is, $x = 1^g, 10^g(10^g)100^g$. The RENO 1 manuscript tabulates sn u for $k^2 = 0(.01)1$, $u/K = 0(.01)1$. BYERLY 1 gives sn u, cn u, dn u to 3D for $k^2 = \frac{1}{2}$, $u = .05(.1)1.85$.

MILNE-THOMSON 3 tabulates sn u, cn u, dn u to 5D, with first differences in u, for $k^2 = 0(.1)1$ and for u at interval .01 from 0 to a limit 2 or more, depending on k^2, but greater than K except at $k^2 = 1$.

F. W. NEWMAN 1 (p. 126), 2 (p. 131) gives $-\log \mathrm{dn}\, \frac{1}{2}K$ to 16D for $\rho = \pi K'/2K = -\frac{1}{2}\ln q = 1(.1)6.4$.

BERGMANN 1 tabulates sn $(u + iv)$ to 2–3D for $\theta = 15°, 20°, 75°$, $u = 0(K/10)K$, $v = 0(K'/10)K'$.

On coefficients in the Fourier expansion of sn u cn u, see Section VII–7 (GLAISHER 1).

S. A. E. WAR ENGINEERING BOARD 1 gives (p. 346) γ to 5D of 1° and sin γ to 7D, where $\sin \frac{1}{2}\gamma = \sin \frac{1}{2}\gamma_0 \,\mathrm{sn}\, u$, where $\gamma_0 = 2\theta = 10°(10°)160°$ and $\tau = \pi u/2K = 0(10°)90°$. More specialized tables of a similar kind follow. See *MTAC*, v. 2, p. 121–122, 1946.

Section XII. WEIERSTRASSIAN FUNCTIONS

Accounts of the Weierstrassian functions are to be found in various texts, for instance in WHITTAKER & WATSON, *Modern Analysis*. Tables relating to such functions are not numerous. WEIERSTRASS & SCHWARZ 1 provides a good collection of formulae and theorems, and on p. 7 tabulates coefficients in the expansion of the sigma function $\sigma(u)$. If

$$\sigma(u) = \sum_{m,n} a_{mn} (\tfrac{1}{2}g_2)^m (2g_3)^n u^{4m+6n+1}/(4m + 6n + 1)!\ (m, n = 0, 1, 2, \cdots)$$

the table gives the (integral) values of a_{mn} for the 33 combinations of m and n for which $4m + 6n + 1 \leqslant 35$. A number of functions are expanded to a considerable length in the lists in TANNERY & MOLK 1, v. 4, p. 88f.

Several Weierstrassian formulae for evaluating half-periods and incomplete integrals (WEIERSTRASS & SCHWARZ 1, p. 67–71) involve quantities $\mathfrak{L}_{0,1}$, $\mathfrak{L}_{0,2}$, $\mathfrak{L}_{0,3}$ defined by omitting from the series

$$\mathfrak{L}_0 = 1 + (\tfrac{1}{2})^2 l^4 + \left(\frac{1.3}{2.4}\right)^2 l^8 + \left(\frac{1.3.5}{2.4.6}\right)^2 l^{12} + \cdots$$

its first, first two, and first three terms respectively. The three quantities are tabulated, when appreciable to the number of decimals given (10 or more), with first differences in the case of $\mathfrak{L}_{0,1}$, in NAGAOKA & SAKURAI 1, p. 56, for $k^2 = 0(.001).5$, where $l = (1 - \sqrt{k'})/(1 + \sqrt{k'})$.

GREENHILL published two tables, both calculated by A. G. HADCOCK, relating to the Weierstrassian function $\wp(u)$ for which

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3$$

in the "equianharmonic" case $g_2 = 0$, the discriminant $g_2^3 - 27g_3^2$ being negative and consequently two roots of the cubic complex.

GREENHILL 1 tabulates $\wp(u)$ to 5–7S when $g_3 = 4$, for arguments $u = r\omega_2/180$ and $u = r\omega_2'/180$, where $r = 0(1)180$ in both cases. Here ω_2 is the real half-period, and $\omega_2' = \omega_1 - \omega_1 = i\omega_2\sqrt{3}$. The corresponding Jacobian modulus is sin 15°, and $\omega_2 = K/3^{\frac{1}{2}}$. The last figures are very inaccurate (see Part III).

GREENHILL 2 tabulates $\wp'(u)$, $\wp(u)$, $\zeta(u)$ and $\sigma(u)$ to about 5D when $g_3 = 1$, for $u = r\omega_2/180$, $r = 0(1)240$. The Jacobian modulus is still sin 15°, and the real half-period ω_2 is now

$$\omega_2 = 2^{\frac{1}{2}}K/3^{\frac{1}{4}} = \tfrac{1}{4}\{\Gamma(\tfrac{1}{4})\}^2/[3^{\frac{1}{4}}\Gamma(\tfrac{3}{4})] = 1.52995.$$

The values of $\wp(u)$ are $2^{-\frac{1}{2}}$ times those in GREENHILL 1. The table is reproduced in GOSSOT 1, and to not more than 4D in JAHNKE & EMDE 1.

In the case when g_2, g_3 are real and the discriminant $g_2^3 - 27g_3^2$ is positive, so that the cubic has real roots, I have encountered no tables of $\wp(u)$. But owing to the occurrence of this case of real roots in Gauss's elliptic ring method of calculating secular perturbations of planets, there have been several tables from astronomical sources for calculating the quantities ω and η, in terms of which various real complete integrals involving $\sqrt{S} = \sqrt{(4s^3 - g_2 s - g_3)}$ may be expressed.

Supposing the roots of the cubic in the order $e_1 > e_2 > e_3$, and all surds positive, we have

$$\omega = \omega_1 = \int_{e_3}^{e_2} \frac{ds}{\sqrt{S}} = \int_{e_1}^{\infty} \frac{ds}{\sqrt{S}},$$

$$\eta = \eta_1 = -\int_{e_3}^{e_2} \frac{s\,ds}{\sqrt{S}}.$$

The expressions in terms of complete elliptic integrals of modulus given by $k^2 = (e_2 - e_3)/(e_1 - e_3)$ are

$$\omega = K/\sqrt{(e_1 - e_3)}, \qquad \eta = \sqrt{(e_1 - e_3)}\{E - \tfrac{1}{3}(2 - k^2)K\},$$

but it was pointed out by H. BRUNS[1] that ω and η may be evaluated in terms of hypergeometric functions, without first finding the roots e_1, e_2, e_3. If an absolute invariant g is defined by

$$g = \frac{g_2^3}{27g_3^2} = \frac{4(1 - k^2k'^2)^3}{\{(2 - k^2)(1 + k^2)(1 - 2k^2)\}^2},$$

and $x = 1 - g^{-1}$, then, if $g_3 \geqslant 0$,

$$\omega(12g_2)^{\frac{1}{2}}/\pi = F_\omega = F(\tfrac{1}{12}, \tfrac{5}{12}, 1, x),$$
$$12\eta(12g_2)^{-\frac{1}{2}}/\pi = F_\eta = F(-\tfrac{1}{12}, \tfrac{7}{12}, 1, x).$$

ARNDT 1 tabulates F_ω and F_η to 7D with Δ for $x = 0(.001)1$. Interpolation is difficult when $x > .98$; ARNDT recommends instead the use of formulae involving four hypergeometric series, for which he tabulates coefficients of powers of $1 - x$.

If a different absolute invariant γ is defined by

$$\cos 3\gamma = g_3\sqrt{(27/g_2^3)}, \qquad 0 \leqslant 3\gamma \leqslant 180°,$$

$\cos 3\gamma$ having the same sign as g_3, then whichever this sign is, we have[2]

$$F_\omega = F(\tfrac{1}{6}, \tfrac{5}{6}, 1, \sin^2 \tfrac{3}{2}\gamma) = P_{-\frac{1}{2}}(\cos 3\gamma),$$
$$F_\eta = F(-\tfrac{1}{6}, \tfrac{7}{6}, 1, \sin^2 \tfrac{3}{2}\gamma) = P_{\frac{1}{2}}(\cos 3\gamma),$$

P as usual denoting a Legendre function. WITT 3 supplements ARNDT 1 by tabulating F_ω and F_η to 8D with Δ for $\xi = .4(.001).5$, where

$$\xi = \tfrac{1}{2}\{1 - \sqrt{(1 - x)}\} = \tfrac{1}{2}(1 - g^{-\frac{1}{2}}) = \sin^2 \tfrac{3}{2}\gamma.$$

The range in ξ corresponds to $.96 \leqslant x \leqslant 1$, the region of difficulty in ARNDT 1. WITT 3 tabulates also, logarithmically, coefficients in continued fractions for F_ω and F_η.

The tables of ARNDT 1 give ω_1 and η_1 only when $g_3 \geqslant 0$, corresponding to $0 \leqslant 3\gamma \leqslant 90°$, and WITT 3 does not extend the range. When $g_3 < 0$, they may be used to determine ω_2 and η_2. For, connected with a given cubic expression, $S = 4s^3 - g_2s - g_3$, having real g_2 and g_3 such that $g_2^3 > 27g_3^2$, there are four positive quantities, ω, η and what I propose here to call ω^*, η^*, such that

$$\omega_1 = \omega, \qquad \omega_2 = i\omega^*, \qquad \eta_1 = \eta, \qquad \eta_2 = -i\eta^*, \qquad \eta\omega^* + \eta^*\omega = \tfrac{1}{2}\pi.$$

ω and η are as defined above, while

$$\omega^* = \int_{e_3}^{e_1} \frac{ds}{\sqrt{(-S)}} = \int_{-\infty}^{e_3} \frac{ds}{\sqrt{(-S)}}, \qquad \eta^* = \int_{e_3}^{e_1} \frac{s\,ds}{\sqrt{(-S)}},$$

all square roots being taken positively. For given g_3, we have

$$\omega^*(g_3) = \omega(-g_3), \qquad \eta^*(g_3) = \eta(-g_3).$$

Thus it is immaterial whether we consider two functions, say ω and η, in the whole range $0 \leqslant 3\gamma \leqslant 180°$, or four functions in half the range, say $0 \leqslant 3\gamma \leqslant 90°$. This is similar to the well-known choice between quadrantal and semi-quadrantal tabulation of the trigonometric functions or of the complete integrals K, E, K', E'. However the matter is regarded, the tables of ARNDT 1 give only half the desirable information, the sign of g_3 being lost in the argument $x = 1 - 27g_3^2/g_2^3$.

INNES used an absolute invariant ι (iota), sometimes with numerical suffix, identical with 3γ above. As ι is an inconvenient symbol, and the angle γ (rather than thrice it) is the simplest quantity to use in writing down the trigonometric solution of the cubic, I shall describe the INNES-ROBBINS-MERFIELD tables in the γ notation. These tables between them cover the whole range $0 \leqslant 3\gamma \leqslant 180°$, and so form a complete set of tables for calculating ω, η, ω^*, η^* for any γ. The formulae for this purpose are set out in INNES 4. They involve the hypergeometric series

$$F(-\tfrac{1}{6}, \tfrac{7}{6}, 2, \sin^2 \tfrac{3}{2}\gamma) \qquad \text{and} \qquad F(\tfrac{1}{6}, \tfrac{5}{6}, 2, \sin^2 \tfrac{3}{2}\gamma),$$

which, unlike the series for $P_{\pm\frac{1}{3}}(\cos 3\gamma)$, remain finite (and in fact vary little) over the whole range $0 \leqslant 3\gamma \leqslant 180°$. Logarithms of these two hypergeometric functions are tabulated to 7D with Δ^2 for $3\gamma = 0(1°)90°$ in both ROBBINS 1 and INNES 4, and for $3\gamma = 90°(1°)180°$ in MERFIELD 2. Logarithms of the first 20 coefficients in each series are given to 8D in MERFIELD 1.

If the four quantities ω, η, ω^*, η^* are to be calculated for given g_2, g_3, whichever the sign of g_3, two hypergeometric functions are taken from the INNES-ROBBINS table, and two from the table of MERFIELD 2; the latter is an important complement to the former. A small illustrative table of $2\omega/\pi$, $2\eta/\pi$, etc. is given for 16 values of γ in INNES 4 (p. 363). This interesting table covers the whole range, but the use of either γ (or 3γ) or $g_3\sqrt{(27/g_2^3)}$ as equidistant argument would have been better; the table would have been symmetrical, and moreover it is curious to see the roots of the cubic figuring as primary arguments when the work aims at avoiding their determination.

The above tables are astronomically related to those of HILL 1, CALLANDREAU 1 and INNES 1. Those of HILL 1, having $\theta = \sin^{-1} k$ as argument, have been mentioned in Section VII–5. CALLANDREAU 1 and INNES 1 are too specialized to be of much general mathematical interest, and all three may probably be regarded as superseded by the later work described above. Discussions of the various methods from an astronomical point of view are given in INNES 3 and by E. DOOLITTLE.[3]

[1] H. BRUNS, *Ueber die Perioden der elliptischen Integrale erster und zweiter Gattung*, Dorpat, 1875. (I have seen only the reprint in *Math. Ann.*, v. 27, 1886, p. 234–252. It is here stated that this reprint is "einer im Jahre 1875 zum Doctorjubiläum von V. J. Buniakowsky von der physiko-mathematischen Facultät der Universität Dorpat dargebrachten Festschrift.")

[2] H. C. PLUMMER, *An Introductory Treatise on Dynamical Astronomy*, Cambridge, 1918, p. 214.

[3] E. DOOLITTLE, *The Secular Variations of the Elements of the Orbits of the Four Inner Planets*, Amer. Phil. Soc., *Trans.*, n.s., v. 22, part 2, Philadelphia, 1912.

Section XIII. DIXON FUNCTIONS

A theory of elliptic functions based upon the cubic curve $x^3 + y^3 - 3\alpha xy = 1$ was developed by A. C. DIXON.[1] The case $\alpha = 0$, which corresponds to the "equianharmonic" case of Weierstrassian functions, has been used in O. S. ADAMS 1 in calculations concerning map projections. When $\alpha = 0$, functions sm u and cm u are defined by

$$x = \operatorname{sm} u, \qquad y = \operatorname{cm} u, \qquad \operatorname{sm}^3 u + \operatorname{cm}^3 u = 1,$$

$$u = \int_0^x (1 - x^3)^{-\frac{1}{3}} dx = \int_y^1 (1 - y^3)^{-\frac{1}{3}} dy,$$

and a real period 3λ is given by

$$\lambda = \int_0^1 (1 - x^3)^{-\frac{1}{3}} dx = \tfrac{1}{3}[\Gamma(\tfrac{1}{3})]^3/\Gamma(\tfrac{2}{3}) = 1.76663\ 875.$$

ADAMS 1 writes K instead of λ, but DIXON's notation is more convenient here.

The connection with the equianharmonic Weierstrassian functions tabulated in GREENHILL 2 and JAHNKE & EMDE 1 (see Section XII) is

$$\operatorname{sm} u = \frac{2\sqrt{3}\wp(u/\sqrt{3})}{\sqrt{3} - \wp'(u/\sqrt{3})}, \qquad \operatorname{cm} u = \frac{\wp'(u/\sqrt{3}) + \sqrt{3}}{\wp'(u/\sqrt{3}) - \sqrt{3}}.$$

$u = \lambda$ corresponds to $r = 120$ in these Weierstrassian tables, and ADAMS 1 constructs his table with arguments $u = 0(\lambda/120)\lambda$. He gives sm u, cm u and sm u/cm u to 4D. In addition, he gives 1 − cm u to 8–5D for $u = 0(\lambda/120)\lambda/3$. Values in the remaining two-thirds of the period are given by

$$\text{sm}(\lambda + u) = 1/\text{cm}\, u, \qquad \text{cm}(\lambda + u) = -\text{sm}\, u/\text{cm}\, u,$$
$$\text{sm}(2\lambda + u) = -\text{cm}\, u/\text{sm}\, u, \qquad \text{cm}(2\lambda + u) = 1/\text{sm}\, u.$$

[1] A. C. DIXON, *Quart. Jn. of Math.*, v. 24, 1890, p. 167–233.

PART II

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PART III

ERRORS IN ELLIPTIC TABLES

NOTE. Absence of a table from this part does not mean that it contains no errors. I have included all information pointed out by others and known to me, but Part III describes mainly my own investigations. My program related only to single-entry tables, mainly of a

basic and many-figure character. Inclusion of an author in **Part III** does not mean that all his elliptic tables have been examined; only those mentioned have been examined. I have checked no double-entry tables whatever; but for the convenience of the user I have placed together under LEGENDRE the known errors in his great double-entry tables of $F(\theta, \phi)$ and $E(\theta, \phi)$. Anyone using one of the many abbreviations of these double-entry tables should see whether they are affected by the errors mentioned under LEGENDRE.

AIREY 1.

(α) K_1, K_2, $k'^2 = [0(.00001).00010; 13D]$.

Differencing shows freedom from gross error, in both function values and second differences. I have also confirmed the values at .00010 by direct calculation. In most of these tables, Airey gives smoothly-varying second differences (presumably rounded from values computed with extra decimals).

(β) K_1, K_2, $k'^2 = [0(.0001).0010; 12D]$.

Differencing shows complete freedom from gross error. I have also confirmed the values at .0010 by direct computation.

(γ) K_1, K_2, $k'^2 = [0(.001).170; 10D]$.

There is a gross error in K_2 at $k'^2 = .014$; *for* 352, *read* 353. When this is corrected, both K_1 and K_2 difference well. Airey's smoothed second differences agree well with the second differences of his function values; the two quantities rarely differ by two final units.

(δ) E_1, E_2, $k'^2 = [0(.00001).00010; 12D]$.

At $k'^2 = .00010$, E_1 *should end in* 75, *not* 57. Apart from this, differencing shows complete freedom from gross error. At .00010 I have also confirmed E_1 (as corrected) and E_2 by direct computation.

(ϵ) E_1, E_2, $k'^2 = [0(.0001).0010; 11D]$.

Differencing shows complete freedom from gross error. I have also confirmed nearly half the values by direct computation.

(ζ) E_1, E_2, $k'^2 = [0(.001).100; 10D]$.

Differencing shows complete freedom from gross error, Airey's second differences again agreeing well with the second differences of his printed values. But recalculation with extra decimals for $k'^2 = .001(.001).010$, while it confirms the values of E_2, shows that the values of E_1 for $k'^2 = .003(.001).009$ average about a unit too low, the deduced values of E being (since E_1 has to be multiplied by about 4) too low by from 2 to 5 units of the tenth decimal. The errors cannot be due to Airey's having used incorrect coefficients in his power series, for his E_1 is correct for much higher values of k'^2, for instance, for $k'^2 = .1$ at the end of the table.

BERTRAND 1.

(α) $\log q$, $\theta = [0(5')90°; 5D]$.

I have compared this table with MEISSEL's 8D table. There are only three errors in MEISSEL 1 which could affect Bertrand's abridgment; all three (marked M below) occur in Bertrand, and are proof that Bertrand's table was extracted without acknowledgment from MEISSEL'S. In addition, BERTRAND (or his collaborator THOMAN) has rounded MEISSEL'S value incorrectly in two cases. At 8°45′ and 51°15′ MEISSEL'S values end in 500, but I have not thought it necessary to investigate more closely.

	θ	*D*	*For*	*Read*
M	17°20′	4	0	1
M	46 25	5	1	2
	51 20	5	3	4
	55 0	5	1	2
M	65 5	5	4	5

(β) $\log \vartheta_3$ [called $\log \Theta(\omega)$], $\theta = [0(30')90°; 9D]$.

The function tabulated is $\frac{1}{2}\log(2K/\pi)$. I have recalculated the table, using 12D values of $\log K$ from LEGENDRE (slightly corrected where necessary). There are the following errors:

θ	D	For	Read	θ	D	For	Read
6°30′	7	1	6	52°30′	4	6	5
7 30	9	1	7	67 30	0	9	0
8 0	8–9	81	08	69 30	9	1	2
41 0	9	4	3	78 0	9	0	1
51 30	9	0	1	82 30	9	6	7

DALE 1.

(α) K, E, $\theta = [0(1°)90°; 5D]$.

There is one rounding error; in K at 88°, *for final* 1, *read* 2.

DWIGHT 2.

(α) $K/\ln(4 \sec \theta)$, $\theta = [86°(1')90°; 7 \text{ or more D}]$.

Differencing shows that the table is free from gross error up to at least 89°50′, and the values at 89°50′(1′)90° I have confirmed by recalculation. Dwight's differences agree with his function values.

(β) K, $\theta = [86°(1')90°; 5D]$.

Differencing shows that the table is free from gross error up to at least 89°50′, and the values at 89°50′(1′)90° I have confirmed by recalculation. Dwight's differences agree with his function values. Comparison of Dwight's values at $\theta = 86°(6')90°$ with antilogarithms of LEGENDRE's log K shows all such values correct, except for one very fine rounding error; at $\theta = 87°24'$, *for* 4.48114, *read* 4.48115. There may be other unimportant rounding errors, but both tables are evidently of high accuracy.

GAUSS 2

The argument in Gauss's table on p. 403 will be called γ. If θ is the usual modular angle, $\gamma = 180° - 2\theta$. Thus Gauss's $\gamma = 1°(1°)180°$ is equivalent to $\theta = 89°.5(-0°.5)0$.

(α) $M = \pi/2K$, $\gamma = [1°(1°)180°; 7D]$.

There are six errors which are not merely rounding errors:

γ	θ	D	For	Read
2°	89°.0	5–7	201	198
13	83 .5	7	4	5
41	69 .5	7	4	6
82	49 .0	7	1	2
156	12 .0	7	8	6
179	0 .5	6	0	1

There are also some 25 rounding errors.

(β) $\log M$, $\gamma = [1°(1°)180°; 7D]$.

There are five errors which are not merely rounding errors:

γ	θ	D	For	Read
2°	89°.0	6–7	80	75
7	86 .5	7	1	0
13	83 .5	5–7	699	701
41	69 .5	7	2	3
45	67 .5	7	4	5

There are also 15 rounding errors.

(γ) On a number of errors in the lemniscate constants on p. 413–432, see J. W. WRENCH, JR., *MTAC*, v. 3, p. 201, 1948.

GLAISHER 1.

(α) q, $\theta = [1°(1°)89°; 8D]$.

Before receiving the SPENCELEY tables I had differenced Glaisher's table and found it free from gross error up to at least 80°. I have since compared the two tables, and found Glaisher in complete agreement with the SPENCELEY values rounded to 8D. Glaisher thus appears free from error of any kind.

(β) $\log q$, $\theta = [1°(1°)89°; 8D]$.

Two very slight rounding errors, exactly as in MEISSEL: at 33°, *for* 819, *read* 820, and at 42°, *for* 332, *read* 333.

I have not examined the other tables.

GOSSOT, see GREENHILL 2.

GREENHILL 1.

(α) $s = \wp(r\omega_3/180)$, $S = \wp(r\omega_2'/180)$, $r = 0(1)180$, 5 or more figures, case $\wp'^2 = 4\wp^3 - 4$.

I have not examined the table in detail, but it may be useful to warn that the final digits are demonstrably unreliable. The values near the end of the table should be derivable from those near the beginning by the relations

$$s(180 - r) - 1 = 3/[s(r) - 1], \quad S(180 - r) - 1 = 3/[S(r) - 1],$$

but some of Greenhill's values do not satisfy this test fully. The table could easily be recalculated, using HIPPISLEY or SPENCELEY at $\theta = 15°$ and 75°, since for $0 \leqslant r \leqslant 90$

$$s(r) = 1 + \sqrt{3}[D(r) + B(r)]/[D(r) - B(r)] = 1 + \sqrt{3}\cot^2 \tfrac{1}{2}\phi \qquad (\theta = 15°)$$
$$S(r) = 1 - \sqrt{3}[D(r) + B(r)]/[D(r) - B(r)] = 1 - \sqrt{3}\cot^2 \tfrac{1}{2}\phi \qquad (\theta = 75°)$$

with obvious modifications when $90 \leqslant r \leqslant 180$.

GREENHILL 2.

(α) $\wp(r\omega_3/180)$, $\wp'(r\omega_3/180)$, $\zeta(r\omega_3/180)$, $\sigma(r\omega_3/180)$, $r = 0(1)240$, about 5D, case $\wp'^2 = 4\wp^3 - 1$.

I have not examined the tables for errors. Two errors were pointed out in O. S. ADAMS, and one by MILLER:

Function	r	*For*	*Read*
$\wp(u)$	3	1468.820	1537.9625
$\wp'(u)$	35	−73.4302	−75.9603
$\sigma(u)$	123	1.0441	1.0438

See *MTAC*, v. 1, p. 109, 1943, and v. 1, p. 395, 1945. The errors occur also in the reprints in GOSSOT and in JAHNKE & EMDE 1_1–1_3, but are corrected in JAHNKE & EMDE 1_4.

HANCOCK 1.

(α) K, E, $\theta = [0(1°)70°(30')80°(12')89°(6')90°; 5D]$.

With one exception, the table is accurately copied from LÉVY 1, none of the errors (which I list under LÉVY) being corrected. The additional error introduced by Hancock is in E at 89°54'; in the fourth decimal, *for* 1, *read* 0.

HAYASHI 1.

This contains a great variety of elliptic tables with k^2 as argument. I have examined only those tables mentioned below, which appeared to me to be the most important. The volume was published in 1930, and the *Berichtigungen* sheet of 1932 will be lacking in copies bought before its appearance. Thus I have thought it useful to include in the lists of corrections those given by Hayashi himself in relation to the tables which I have examined, especially since Hayashi's corrections require occasional emendation. I have not reprinted his corrections relating to unexamined tables.

(α) K, $k^2 = [0(.001)1;10\text{–}12D]$.

There are the following errors:

p.	k^2	D	*For*	*Read*
72	.999	8–9	06	60
72	.013	8–9	68	86
72	.987	11–12	53	47
73	.939	12	4	1
73	.095	0	0	1
74	.877	11–12	61	57
77	.737	9–10	56	65
81	.493	9	4	9

Those at .999, .737 and .493 were corrected in the 1932 *Berichtigungen* sheet, which also gives a false and unnecessary correction at .956. The obvious error at .095 was pointed out by SAMOĬLOVA-ĬAKHONTOVA, and the error at .013 by KAPLAN. See also *MTAC*, v. 2, p. 127, 1946, and COMRIE, *Math. Rev.*, v. 7, 1946, p. 485. As far as I am aware, the errors of 6, 4 and 3 final units at .987, .877 and .939 respectively have not previously been published.

When the above corrections are made, differencing (up to twentieth differences at the end) shows freedom from gross error up to at least $k^2 = .984$. The last 15 values, for $k^2 = .985(.001).999$, I have recomputed to about 14 decimals; except for the errors at .987 and .999, mentioned above, Hayashi is completely correct.

(β) $\log q$, $k^2 = [0(.001)1;10D]$.

The first four of the following corrections were given in Hayashi's 1932 *Berichtigungen* sheet (the argument .232 being misprinted .223), and the fifth in SAMOĬLOVA-ĬAKHONTOVA 1.

p.	k^2	D	*For*	*Read*
73	.050	3	3	5
74	.874	10	8	7
76	.232	4–5	62	26
78	.688	10	8	9
81	.503	1–2	36	63

When these corrections are made, differencing shows freedom from gross error, except that some ten or twelve values at each end of the table cannot be completely checked by differencing. I have, however, recomputed 28 values, for $k^2 = .001(.001).014$ and $k^2 = .986(.001).999$, to about 13D, and find no important error (my values suggest rounding errors at $k^2 = .010, .012, .013$, where *for final* 8, 1, 7, I find 7, 0, 6 respectively).

(γ) q, $k^2 = [0(.001).5;8D]$.

Hayashi's 1932 *Berichtigungen* sheet gives the following corrections:

p.	k^2	D	*For*	*Read*
94	.279	8	7	8
98	.395	5	4	6
102	.482	3	0	1

When these corrections are made, differencing shows freedom from gross error. The last decimal appears to be good, which is not surprising, as Hayashi also tabulates 10D logarithms; see (β) above. Every tenth value has been compared with the 10D value given in MILNE-THOMSON 1; this failed to disclose any rounding errors in Hayashi at $k^2 = 0(.01).5$, and computation shows that Hayashi is correctly rounded at $k^2 = .47$, where MILNE-THOMSON ends in 50.

(δ) q^1, $k^2 = [0(.001).5;8D]$.

Hayashi's 1932 *Berichtigungen* sheet gives the following corrections (that at .118 is illegible, but should be as below):

p.	k^2	D	*For*	*Read*
84	.008	8	7	8
84	.024	3–5	648	739
86	.068	3–8	816360	756984
86	.077	4	1	0
86	.081	8	6	3
88	.118	4–8	58203	63000
88	.125	7–8	70	66
88	.142	5–7	928	784
90	.161	6–8	493	049
92	.236	0–5	011	0.36011
94	.256	6–8	728	073
96	.315	4–5	60	06

The above leaves the following errors uncorrected:

p.	k^2	D	*For*	*Read*
84	.028	5	6	5
88	.120	7	2	0
90	.193	0–5	022	0.34022

When all the above corrections have been made, differencing shows freedom from gross error, except that about 8 values at the beginning of the table cannot be checked to the full 8 decimals by differencing. I have, however, recomputed the first 14 values, for $k^2 = .001(.001).014$, to 10 decimals, and confirmed Hayashi's results completely.

(ε) $\pi\vartheta_1'$, $k^2 = [0(.001).5;8D]$.

The function is called $\vartheta_1'(0)$ in Hayashi. His 1932 *Berichtigungen* sheet gives the following corrections:

p.	k^2	D	*For*	*Read*
84	.049	3	1	7
92	.207	8	6	5
94	.292	5	1	2
96	.312	5	4	6
96	.342	4	2	1
102	.498	5	3	8

This leaves one gross error uncorrected: p. 102, $k^2 = .465$, 5D, *for* 8, *read* 9.

When all these corrections have been applied, differencing shows freedom from gross error, except that the first 9 values cannot be fully checked by differencing. I have, however, recomputed the first 10 values, for $k^2 = .001(.001).010$, to 9 or more decimals, and confirmed Hayashi's values (apart from one or two possible slight rounding errors).

(ζ) ϑ_2, $k^2 = [0(.001).5;8D]$.

Hayashi's 1932 *Berichtigungen* sheet gives the following corrections:

p.	k^2	D	*For*	*Read*
97	.328	5	9	4
101	.442	4–5	37	73

When these are applied, differencing shows that no gross error remains, except that the first 8 values cannot be fully checked by differencing. I have, however, recomputed the first 10 values, for $k^2 = .001(.001).010$, to about 10 decimals, and confirmed Hayashi's results.

(η) ϑ_3, $k^2 = [0(.001).5;8D]$.

Hayashi's 1932 *Berichtigungen* sheet gives the following correction:

p. 85, $k^2 = .026$, 7D, *for* 8, *read* 9.

When this is applied, differencing shows freedom from gross error throughout the table.

(θ) ϑ_4, $k^2 = [0(.001).5;8D]$.

This is called $\vartheta_0(0)$ in Hayashi. Differencing reveals one gross error:

p. 88, $k^2 = .112$, 6D, *for* 2, *read* 3.

Otherwise there are no gross errors.

HAYASHI 3.

(α) E, $k^2 = [0(.001)1;10D]$.

There are known errors as follows (KAPLAN; also COMRIE, *Math. Rev.*, v. 7, 1946, p. 485):

p.	k^2	D	*For*	*Read*
62	.052	9	4	5
62	.939	9	4	3
62	.936	9	9	8
63	.201	3–4	65	86
63	.732	10	6	8
64	.668	0	0	1

At .669, the first two digits of the argument are missing. SAMOĬLOVA-ĨAKHONTOVA gave a faulty correction at .201, misprinting the argument as .151 and failing to correct the fourth decimal.

When the above corrections are made, differencing shows freedom from gross error up to at least $k^2 = .991$. I have recomputed the values for $k^2 = .990(.001).999$, and found only two rounding errors; at $k^2 = .991$, *for final* 1, *read* 0, and at $k^2 = .996$, *for final* 8, *read* 9.

Comparison with my 12D calculations at $k = .1(.1).9$, with SCHMIDT 1 at $k' = .1(.1).9$ and with SPENCELEY 1 at $\theta = 30°, 45°, 60°$ reveals only one very fine rounding error; at $k^2 = .360$, *for final* 5, *read* 4. Slight roughness in differences near $k^2 = .938$ points to at least one other last-figure error, but in general the final digit appears to be good.

HEUMAN 1.

It is not surprising that the 6D tables of $F_0(\alpha)$ and $E_0(\alpha)$, our $2K/\pi$ and $2E/\pi$, are very exact, for Heuman says that he calculated to 10D, from LEGENDRE'S logarithms. As these tables are, as far as I know, the only *natural* tables of, substantially, K and E at interval 0°.1 throughout the quadrant, I have examined them with care.

(α) $F_0(\alpha) = 2K/\pi$, $\alpha = [0(0°.1)90°;6D]$.

Assuming the values at 67°.5 and 76°.8 corrected to 1.527948 and 1.838639, as shown in Heuman's errata, the table is then certainly free from anything like gross error. I have differenced up to 70° and also recomputed to at least 8D the values at 0(0°.5)70°(0°.1)90°. In the values recomputed, I have found only one certain rounding error; at 5°.5, *for final* 8, *read* 9. I have not investigated further cases where my seventh and eighth decimals were 50. All Heuman's differences accurately correspond to his function values (corrected at 67°.5 and 76°.8).

(β) Auxiliary function $G_0(\alpha)$, $\alpha = [65°(0°.1)90°;6D]$.

Differencing shows that the table is free from gross error, and that Heuman's differences accurately correspond to his function values.

(γ) $E_0(\alpha) = 2E/\pi$, $\alpha = [0(0°.1)90°;6D]$.

Differencing shows that the table is free from gross error, and that Heuman's differences accurately correspond to his function values. Here also the rounding to 6D appears to be highly accurate. I have recomputed the values at 0(1°)90° to nine decimals, and find only one very slight rounding error; at 16°, *for final* 7, *read* 6.

(δ) $\Lambda_0(\alpha, \beta)$, $\alpha = 0(1°)90°$, $\beta = 0(1°)90°$, etc., 6D.

I have not examined this important table, as it is of double entry and so falls outside my program. Heuman himself gives two corrections:

p.	α	β	*For*	*Read*
185	71°	15°	94145	84145
196	0°.3	87°	8523	8623

HIPPISLEY 1.

I have not examined the tables for errors. Seven corrections are given in *MTAC*, v. 1, p. 325, 1944.

INNES 2.

(α) $\log q$, $\theta = [0(1°)45°;10D]$.

I have derived 10D logarithms (uncertain by about a unit in the tenth place) of the values of q given in SPENCELEY 1. These never differ from the values of Innes by more than four units in the tenth place. The table is thus free from gross error. Innes did not expect his tenth decimal to be exact.

(β) $\log \log (1/q)$, $\theta = [0(1°)45°;10D]$.

The table is correctly abbreviated from VERHULST 1. The only error in Innes is therefore at 21°; in the ninth and tenth decimals, *for* 23, *read* 17.

(γ) log q, θ = [0(1′)10′,44°50′(1′)45°;8D].

At 44°55′, *for* 380981, *read* 380891. Also on p. 496 are little tables of q and log log $(1/q)$ for the same arguments; these are free from gross error.

JAHNKE & EMDE 1.

I have read only the tables of log q, K, E as functions of θ; and only in the latest edition, which seems the simplest way to contribute to the perfection of future editions. I add a few corrections pointed out by others in various editions. I am concerned here only with tables; on errors in formulae, see *MTAC*, v. 1, p. 108, 198, 391–399.

(α) log q, θ = [0(5′)90°;4D].

The table is abbreviated from BERTRAND's 5D table with the same arguments. It thus descends ultimately from MEISSEL's 8D table.

BOWMAN, *MTAC*, v. 3, p. 41, 1948, has pointed out the following errors in the characteristics of the logarithms in all editions of Jahnke & Emde: at 2°55′, 3°55′, 4°55′, 5°55′, *for* $\bar{5}$, *read* $\bar{4}$; at 37°50′, *for* $\bar{5}$, *read* $\bar{2}$.

All other errors in 1_4 are errors of 1 unit in the fourth place, and were found by reading against MEISSEL.

There are first the following three errors:

	θ	*For*	*Read*
M	17°20′	$\bar{3}.7640$	$\bar{3}.7641$
	38 55	$\bar{2}.4960$	$\bar{2}.4959$
M	65 5	$\bar{1}.0270$	$\bar{1}.0271$

The first and last, marked M, are due ultimately to errors in MEISSEL. The second corresponds to no error in BERTRAND (if all copies agree with mine), but could have been caused by an error in LÉVY.

There are also 56 further errors, all of the same type. BERTRAND's last figure is 5, and Jahnke & Emde wrongly round upwards. In all 56 cases their value should be reduced by one unit of the fourth decimal. The arguments concerned are:

2° 5′	11°10′	17°25′	28°30′	45°10′	53°25′	71° 5′	83°50′
5 25	11 15	18 20	29 15	45 45	63 0	73 5	85 10
6 45	11 40	18 30	34 55	49 20	64 45	76 50	85 40
7 40	12 5	18 55	35 35	49 25	66 0	78 40	86 50
8 15	13 45	19 25	37 0	49 35	68 0	79 20	86 55
9 45	13 50	20 20	41 50	49 40	68 15	80 30	87 30
10 0	15 20	27 25	43 5	50 25	69 10	83 20	89 30

The above are all the errors in 1_4. COMRIE, in his list in *MTAC*, v. 1, p. 392, 1945, gives two errors in 1_1 only; at 64°5′, *for* 4087, *read* 0087, and at 69°55′, *for* 1195, *read* 1159. I observe that both are corrected in the 1923 reprint.

(β) K, E, θ = [0(1°)70°(30′)80°(12′)89°(6′)90°;4D].

The errors surviving in 1_4 are:

Function	θ	*For*	*Read*
K	35° 0′	1.7313	1.7312
	46 0	1.8692	1.8691
	84 12	3.6853	3.6852
	86 48	4.2746	4.2744
	87 24	4.4812	4.4811
	88 12	4.8479	4.8478
E	7 0	1.5650	1.5649
	22 0	1.5142	1.5141
	54 0	1.2682	1.2681
	62 0	1.1921	1.1920

Function	θ °	For	Read
E	66 0	1.1546	1.1545
	67 0	1.1454	1.1453
	69 0	1.1273	1.1272
	81 24	1.0313	1.0314
	86 0	1.0087	1.0086
	88 12	1.0022	1.0021
	89 18	1.0005	1.0004

The following errors, in earlier editions only, are given in *MTAC*, v. 1, p. 198, 1944, from GLAZENAP 1:

Function	θ	For	Read	
K	86°48′	4.2744	4.2746	$1_1, 1_2, 1_3$
	87 36	4.5619	4.5609	$1_1, 1_2, 1_3$
	89 36	6.3504	6.3509	$1_1, 1_2, 1_3$
E	8 0	1.5630	1.5632	$1_1, 1_2$

(γ) $F(\theta, \phi)$, $E(\theta, \phi)$

See the note at the beginning of Part III. Also *MTAC*, v. 1, p. 198, 1944, and v. 1, p. 395, 1945.

(δ) $\wp(u)$, $\wp'(u)$, $\zeta(u)$, $\sigma(u)$, $r = 0(1)240$, case $\wp'^2 = 4\wp^3 - 1$.

See under GREENHILL 2.

MTAC, v. 1, p. 392, 1945, gives, from AIREY, a correction to ω_2 on p. 72 of 1_3 only (correct in the 1923 reprint); *for* 1.53995, *read* 1.52995.

KAPLAN 1.

(α) K, E, $\log k'^2 = [\overline{1}(.005)\overline{2}(.01)\overline{6};10D]$.

Differencing shows the function values in these useful tables to be free from gross error. I have not examined Kaplan's differences.

LEGENDRE (general remarks).

I have had the modern photographic reprints, Legendre 6 (POTIN), 7 (EMDE) and 8 (PEARSON) continuously available. The original publications, Legendre 1–5, I have inspected only in libraries. Legendre 6 and 7 reproduce Legendre 3, while Legendre 8 reproduces Legendre 5. The lists of errors will relate primarily to the reprints on which I have worked. The contents of the reprints are not identical, and in fact natural values of K and E are reproduced only from 3 (in 7), and logarithmic values of K and E only from 5 (in 8). I state whether the errors listed for K and E also occur in 5, but have not otherwise examined K and E in 5; similarly for log K and log E and 3.

Legendre's great double-entry tables of $F(\theta, \phi)$ and $E(\theta, \phi)$ are reproduced in all three reprints, so that it is easy for a modern worker to compare Legendre 3 (through 6 or 7) with Legendre 5 (through 8). Double-entry tables have fallen outside my own program of differencing and recomputation, but in view of the importance of Legendre's double-entry tables, a rearrangement of an errata list due to E. L. KAPLAN is included. Complete checking of the double-entry tables would be such a large task that, as far as I know, it has never been performed, but KAPLAN's list contains many more items than any previous list of errors.

The original tables of Legendre are now so rare that I believe that the reprints alone have now much general working value. Moreover, photographic reproductions of this kind give a guarantee against variations from one copy to another, so that a complete list of errors in one copy would enable all other copies to be completely corrected with certainty. I trust that my investigations concerning the single-entry tables in the reprints will form a useful contribution. It will be evident that much work remains to be done even in this limited sphere, in connection with the tables of log K and log E.

LEGENDRE 3, 5.

Table VI (p. 323–332 of Legendre 3, p. 269–278 of Legendre 5), containing logarithms of the moduli in LANDEN's scale (see **Part I**, Section I), has not been reprinted. The following errors were pointed out by C. J. MERFIELD, *Astron. Jn.*, v. 30, p. 190, 1917, in relation to Legendre 5; I observe that the last two are absent in Legendre 3. I have corrected the last two arguments, which MERFIELD misprinted.

θ	Function	*For*	*Read*
5°.1	$\log c$	5.94887	8.94887
9 .5	$\log b'$	9.99992 96	9.99998 96
34 .0	$\log c$	9.94756	9.74756

LEGENDRE 7.

The tables were reproduced by EMDE from Legendre 3. I have not examined the unimportant first part (p. 338–341) of T.VIII.

(α) K, E, $\theta = [0(1°)90°;12D]$.

The complete integrals in the second part (p. 342–344) of T.VIII are highly important. Before receiving the SPENCELEY tables I had differenced Legendre's values of K and E, and also recomputed these functions for $\theta = 5°(5°)70°(1°)89°$, and for various other arguments where errors were suspected. The following lists should be complete; they are based on SPENCELEY, and I agree with the correction in every case which I have recalculated. Besides the erroneous and corrected digits in the eleventh and twelfth decimals, the corrections (SPENCELEY minus Legendre) are also shown; *r* indicates that the error is merely one of rounding, Legendre's value being within 1 final unit of the true (unrounded) value; +1 and −1 consequently indicate departures of between 1 and 1½ final units from the truth. It will be observed that the maximum correction is 18 units of the twelfth decimal in $K(85°)$, and that the corrections in the table of E are all small.

There are two gross errors in Legendre's differences:

$\Delta^3 E(10°)$, *for* 25070, *read* 15070; $\Delta^3 K(33°)$, *for* 82000, *read* 82080.

Otherwise his differences are consistent with his function values, except for errors in the last two figures of $\Delta^4 E(36°)$, $\Delta^4 E(37°)$, $\Delta^4 E(37°)$, $\Delta K(62°)$, $\Delta K(63°)$, $\Delta K(65°)$.

	K				E		
θ	*For*	*Read*	corr.	θ	*For*	*Read*	corr.
11°	75	76	r	5°	77	78	r
12	27	28	r	23	15	17	+2
24	13	12	r	24	53	54	r
39	04	06	+2	27	43	44	+1
46	31	26	−5	28	26	27	+1
47	84	80	−4	39	85	86	r
48	65	64	−1	40	23	24	r
49	38	36	−2	46	31	30	−1
50	06	05	−1	47	18	17	r
53	35	34	r	48	82	81	r
54	01	05	+4	50	97	98	r
62	81	79	−2	54	64	65	+1
81	43	44	r	55	78	80	+2
83	73	72	−1	66	25	24	r
85	66	84	+18	69	57	58	r
89	25	26	r	73	36	38	+2
				74	74	73	−1
				79	45	44	r

All the above errors, except one in the differences, also appear in the second part (p. 288–290) of T.VIII in Legendre 5. The error in $\Delta^3 K(33°)$ is present neither in Legendre 5 *nor in 6*

copies of Legendre 3 inspected, although Legendre 7 (EMDE) reproduces another copy of Legendre 3! This illustrates the advisability of concentrating research on the reprints.

LEGENDRE 8.

The tables were reproduced by PEARSON from Legendre 5. The extremely important tables of log K and log E will be considered in six parts. From 0 to 15° and from 75° to 90° the function values have 14D, but the differences throughout relate to 12D values. In the 14D portions, I have aimed primarily at finding errors seriously affecting the first 12D, but in the case of log K I have gone further, and the corrections given will turn Legendre's 14D tables into reasonably good 13D tables.

In general, I have not read Legendre's differences systematically against my difference sheets, and the only statement I make concerning them will be found under (β). See also KAPLAN 2, and my comment in RMT 568.

(α) log K, θ = [0(0°.1)15°;14D].

I have differenced the values to the full 14D. The largest error is undoubtedly one of 14 units of the last place at θ = 5°.0. I have recalculated a few values at arguments suggested by the differencing. When the following corrections to the end figures are made, it is probable that no further correction will exceed some four units of the fourteenth decimal.

θ	*For*	*Read*
1°.1	35	30
5 .0	26	40
5 .3	06	11
5 .5	01	05

All the above errors also occur in Legendre 3.

(β) log K, θ = [15°(0°.1)75°;12D].

I have differenced the values. The only gross errors in the function values are: θ = 24°.9, 7D, *for* 2, *read* 3; θ = 30°.9, 0D, *for* 8, *read* 0.

These corrections have previously been pointed out by HEUMAN; the differences correspond to the corrected values, and, as has been pointed out by ARCHIBALD (*MTAC*, v. 2, p. 181, 1946), the errors are absent in Legendre 3. When the corrections are made, the table differences very well; the sixth difference never exceeds 17, so that even the twelfth decimal must be quite good.

I have examined the *first* differences between 15° and 75°, and found one gross error; at 74°.5, *for* 955, *read* 953. There are no other gross errors, but the last digit is one unit out of correspondence with Legendre's function values at 37°.0 and 75°.0. The first of these three errors is absent in Legendre 3.

(γ) log K, θ = [75°(0°.1)90°;14D].

As in (α), function values are to 14D, differences to 12D. Differencing to the full 14D proved the later decimals to be unreliable, although the first 9D are everywhere correct and only in one instance (at 89°.3) is the 10th affected. Attempts to detect errors affecting the twelfth decimal quickly showed that more than differencing would be required, and in order to make a satisfactory first contribution to the correction of the table I have lavished more time on it than on any other table investigated for this *Guide*; yet much more could still be done. I recalculated to 14D (with slight uncertainty in the last place) values of log K for 81 of the 150 arguments, including θ = 75°(0°.5)84°(0°.1)85°.5(0°.5)88°(0°.1)89°.9, and found disagreements of more than three final units in the 27 cases listed below. These 27 cases I have recalculated to 15 or 16D, in order to try to give definitive corrections for them. When the corrections are made, differencing now suffices to show freedom from other than last-figure error up to at least 88°. The list is not very likely to be complete in respect of errors exceeding three final units, but from the difference tables I judge it to include all errors of ten or more final units, and it may even be found ultimately to include all errors of six or more final units. A noteworthy feature of the results is that the errors are largely

in the twelfth decimal, the next two being frequently correct or nearly so. To exhibit this more clearly, the list gives, besides the erroneous and corrected values in the tenth to fourteenth places, the correction (true value minus Legendre). There is clearly an unexpected number of errors approximately multiples of 100; in particular, nine of ten successive values (between 84°.3 and 85°.2) are too small by about two units in the twelfth place. The attempt to correct the table has convinced me (as other computations convinced KARL PEARSON) of the vast labor which Legendre, without calculating machines, must have put into his tables.

θ	*For*	*Read*	corr.
78°.6	19291	19301	+10
79 .1	38162	38189	+27
79 .4	45626	45631	+5
79 .5	34061	34067	+6
80 .1	26294	26289	−5
80 .3	60617	60515	−102
83 .0	63180	63170	−10
83 .2	57234	57269	+35
83 .7	47733	47632	−101
84 .3	92765	92965	+200
84 .4	52700	52900	+200
84 .5	32799	33002	+203
84 .6	87556	87623	+67
84 .7	08925	09130	+205
84 .8	81351	81552	+201
84 .9	67275	67475	+200
85 .0	31833 31823	32033	+200 +210
85 .1	08187	08385	+198
85 .2	09462	09661	+199
85 .3	93083	93116	+33
85 .7	48980	48989	+9
86 .8	54662	51623	−3039
86 .9	34046	34188	+142
87 .1	30193	30198	+5
88 .7	05699	06099	+400
88 .9	76200	76194	−6
89 .3	21710	31710	+10000

All the above errors also occur in Legendre 3.

(δ) $\log E$, $\theta = [0(0°.1)15°;14D]$.

I have differenced the 14D values. The maximum fifth difference is 162, so that, as in (α), no error will seriously affect the twelfth decimal.

(ϵ) $\log E$, $\theta = [15°(0°.1)75°;12D]$.

I have differenced the values. The only gross errors in the function values are:

θ	D	*For*	*Read*
22°.4	12	0	9
34 .4	9	8	0
36 .2	10	7	5
47 .7	11	1	0

The correction at 34°.4 has already been pointed out by HEUMAN; the others have recently been published in KAPLAN 2, and I found them independently. All four also occur in Legendre 3. In all four cases Legendre's differences agree with function values corrected as above, so

that the mistakes are simply misprints. When the corrections have been made, the table differences extremely well; the fifth difference never exceeds 11, and the twelfth decimal must be quite good. In $\Delta \log E$ (37°.0), *for* 181, *read* 151 (MERFIELD, *loc. cit.*).

(ζ) $\log E$, $\theta = [75°(0°.1)90°;14D]$.

Legendre differences to only 12D. Differencing of the full 14D values shows that the table is not reliable to more than 12D. A gross error, pointed out by HEUMAN, is: $\theta = 86°.0$, 9D, *for* 6, *read* 3. Legendre's differences again agree with the corrected value, and, as pointed out by ARCHIBALD (*MTAC*, v. 2, p. 181, 1946), the error is absent in Legendre 3. At $\theta = 84°.4$, a decimal point should replace the letter x. Between 75° and 89°, the differencing shows that values rounded to 12D will be free from gross error, once that at 86°.0 has been corrected. The table deserves detailed investigation of the kind described in (γ) for the corresponding table of $\log K$.

LEGENDRE 6, 7, 8.

I have not examined Legendre's great double-entry tables for errors by differencing or recomputation, but for the convenience of the reader I had, before receiving KAPLAN 2, drawn up lists of the errors stated by BOHLIN, MERFIELD, GLAZENAP, SAMOĬLOVA-ĨAKHONTOVA, HEUMAN, WITT and ARCHIBALD; for references, see RMT 568 (in the present number). As I found KAPLAN 2 to contain all such known errors, and also many more found by KAPLAN, I have, by kind permission, incorporated all KAPLAN's material on p. 18–19 of his paper, as well as his information on p. 15 about $F(76°, \phi)$, into fresh lists, separating errors in $F(\theta, \phi)$ from errors in $E(\theta, \phi)$, and arranging each list in order of increasing θ. The lists below also contain three known errors in $E(\theta, \phi)$ of 1816, absent from PEARSON's reprint of the 1826 values, and hence outside the scope of KAPLAN's list.

There are two versions of the tables; Legendre 3 (1816) is reproduced in Legendre 6 and 7, while Legendre 5 (1826) is reproduced in Legendre 8. Legendre 6 and 7 no doubt reproduce different copies of Legendre 3, but these two reprints give identical values at all arguments mentioned below. An index, 1 or 2, attached to an erroneous value indicates that the error occurs only in the 1816 or 1826 version respectively. The values labeled "obscure" in KAPLAN 2 are so only in Legendre 8, not in the beautifully printed table in Legendre 5, hence I have called them "Obscure[2] reprint."

Owing to repetition of first and last columns and lines, tabular values are given twice if θ is one of 5°(5°)85°, twice if $\phi = 45°$, and four times if both hold. Errors are included in the lists if they occur at all, even if the multiple presentation includes the correct value in another place.

The errors of one or two units given on p. 15 of KAPLAN 2 have not been incorporated; KAPLAN remarks that they are presumably only typical of considerable portions of the double-entry tables.

(α) $F(\theta, \phi)$, $\theta = 0(1°)90°$, $\phi = [0(1°)90°;9\text{–}10D]$.

θ	ϕ	*For*	*Read*
2°	42°	0.73311 09968	0.73311 00968
3	32	0.55858 03004	0.55858 01004
	64	1.11750 64382[2]	1.11750 63482
5	67	Obscure[2] reprint	1.17091 31870
	73	1.27598 58448[2]	1.27598 56448
7	5	0.08729 81052	0.08726 81052
	54	Obscure[2] reprint	0.94421 96925
9	20	0.34293 53662	0.34923 53662
12	89	1.57035 40839	1.57035 40439
14	1	0.01743 33444[2]	0.01745 33444
	6	0.11473 09355[2]	0.10473 09355
20	32	0.56174 72078	0.56174 52078
21	88	1.58584 47725	1.58784 47725
23	90	1.63631 74093	1.63651 74093

θ	ϕ	For	Read
26	89	1.63627 86937	1.63627 85937
27	42	0.74754 01816	0.74574 01816
	59	1.06251 39233	1.06251 29233
	85	1.56840 67133	1.56480 67133
30	35	0.62203 01811[2]	0.62003 01811
31	12	0.20984 42225[3]	0.20984 42245
34	84	1.59518 05343	1.59518 05543
44	80	1.58906 49280	1.59806 49280
46	51	0.95158 7953	0.95157 7953
	90	1.86914 7545	1.86914 7546
57	90	2.08035 8167	2.08035 8067
59	65	1.34195 7808	1.34196 7808
60	60	1.21253 6614	1.21259 6614
64	79	1.84693 8718	1.84793 8718
66	89	2.30100 5177	2.30100 5157
71	61	1.29179 6493	1.29719 6493
74	85	2.39615 7101	2.39615 6101
75	37	0.69130 9761[2]	0.69190 9761
	84–88	Increase by 1×10^{-9}	
76	55	1.13561 6079	1.13561 6100
	56	1.16468 3555	1.16468 3376
	57	1.19438 4925	1.19438 4946
	58	1.22475 3902	1.22475 3923
	59	1.25582 5454	1.25582 5475
	60	1.28763 6943	1.28763 6964
	61	1.32022 8066	1.32022 8088
	62	1.35364 1002	1.35364 1024
	63	1.38792 0552	1.38792 0575
	64	1.42311 4279	1.42311 4302
	65	1.45927 2632	1.45927 2656
	66	1.49644 9058	1.49644 9083
	67	1.53470 0075	1.53470 0101
	68	1.57408 5305	1.57408 5331
	69	1.61466 7446	1.61466 7473
	70	1.65651 2149	1.65651 2176
	71	1.69968 7781	1.69968 7809
	72	1.74426 5022	1.74426 5051
	73	1.79031 6254	1.79031 6284
	74	1.83791 4673	1.83791 4703
	75	1.88713 3051	1.88713 3082
	76	1.93804 2064	1.93804 2095
	77	1.99070 8091	1.99070 8120
	78	2.04519 0371	2.04519 0398
	79	2.10153 7463	2.10153 7488
	80	2.15978 2927	2.15978 2950
	81	2.21994 0244	2.21994 0265
	82	2.28199 7092	2.28199 7111
	83	2.34590 9231	2.34590 9247
	84	2.41159 4470	2.41159 4485
	85	2.47892 7378	2.47892 7391
	86	2.54773 5614	2.54773 5625
	87	2.61779 8834	2.61779 8843
	88	2.68885 1060	2.68885 1066

θ	ϕ	For	Read
	89	2.76058 7039	2.76058 7042
78	90	2.97856 8952	2.97856 8951
79	50	1.00317 2141	1.00317 2041
83	16	0.28289 5780	0.28289 7580
	31	0.55907 5915[2]	0.56907 5915
	74	1.92525 6351	1.92515 6351
84	18	0.31936 7699	0.31939 7699
86	86	3.17204 1744	3.17030 9981
87	49	0.98238 1667	0.98328 1667
	87	3.45644 5172	3.45667 6092
88	4	0.06936 9880[2]	0.06986 9880
90	31	0.55956 2733[2]	0.56956 2733

(β) $E(\theta, \phi)$, $\theta = 0(1°)90°$, $\phi = [0(1°)90°;9\text{–}10D]$.

θ	ϕ	For	Read
4°	51°	0.88952 98250	0.88962 98250
6	44	0.76721 15513	0.76721 15313
8	90	1.56296 22295	1.56316 22295
10	5	0.00237 31277[2]	0.08726 31277
	41	0.71391 98746	0.71391 97846
	72	Obscure[2] reprint	1.24934 56503
11	7	0.12215 20142[2]	0.12216 20142
15	54	0.93450 23938[2]	0.93460 23938
16	45	0.77994 47234[2]	0.77994 47334
20	14	0.24406 47559	0.24406 47569
	50	Obscure[2] reprint	0.86142 06177
22	78	1.31072 85082	1.31972 85082
23	62	Obscure[2] reprint	1.05610 64197
24	20	0.34791 80029	0.34791 80019
	40	0.68953 08235	0.68953 08335
33	48	0.81181 56955	0.81181 56965
34	41	0.69797 33784	0.69797 33384
35	45	0.76188 30381	0.76128 30381
36	45	0.76003 72623	0.76003 72625
40	69	1.10811 86991	1.10811 86891
42	27	0.46376 10915	0.46366 10915
45	73	1.13785 83451	1.13785 43451
51	23	0.33502 9979[1]	0.39502 9979
64	90	1.17317 9382	1.17317 9383
75	45	0.71289 3043	0.71289 3045
79	32	0.53101 7384	0.53101 1384
82	14	0.24195 9269[1]	0.24196 9269
85	21	Obscure[2] reprint	0.35843 1155
86	2	0.03489 9351[1]	0.03489 9531
	6	0.10462 9396[2]	0.10452 9396

Lévy 1

(α) K, $\theta = [0(1°)70°(30')80°(12')89°(6')90°;5D]$.

Comparison with an 8D table, partly deduced from Legendre and partly computed independently, shows the following errors:

θ	D	For	Read
23° 0′	4	3	5
70 30	4–5	29	30
71 30	4–5	90	89

θ	D	*For*	*Read*
74 30	5	2	3
82 48	5	2	3
83 12	5	4	5
85 36	5	7	6
86 12	5	6	7
87 36	3–5	190	091
89 36	4–5	38	89

(β) $E, \theta = [0(1°)70°(30')80°(12')89°(6')90°;5D]$.

Comparison with an accurate table deduced from LEGENDRE shows the following errors:

θ	D	*For*	*Read*
8° 0′	3–4	29	31
81 24	4	2	3
89 18	4	4	3
89 54	4	1	0

(γ) $\log q, \theta = [0(5')90°;5D]$.

This table was copied with acknowledgment from BERTRAND, who abbreviated from MEISSEL without acknowledgment. In addition to the errors listed under BERTRAND (α), Lévy has seven more through faulty copying:

θ	D	*For*	*Read*
1°30′	2	8	3
2 55	0	$\overline{5}$	$\overline{4}$
33 45	1	4	3
38 55	5	5	3
59 35	1	7	9
64 5	1	4	0
69 55	3–4	95	59

MEISSEL 1.

(α) $\log q, \theta = [0(1')90°;8D]$.

This is the largest single-entry table relating to elliptic functions in existence. It is not so well known as many slighter works, and copies are somewhat rare. Lack of acknowledgment, especially in BERTRAND 1, has hindered recognition of the fact that it is the source of several better known abbreviations. Three errors obviously derived from Meissel show that the 5D table at interval 5′ in BERTRAND 1 was extracted from Meissel 1. Bertrand said (p. xii) that his tables were due to F. THOMAN. Bertrand's book was published in 1870; if in the later part of the year (on which point I lack information), it is possible that the Franco-Prussian war made inopportune a candid acknowledgment of the German origin of the table of $\log q$. Subsequent copyists and abbreviators (LÉVY, POTIN, even JAHNKE & EMDE) have followed BERTRAND, not Meissel.

I have not differenced Meissel's table completely, but in order to check several other tables I have differenced (i) the values at 0(6′)90°, (ii) the values at 0(5′)90°. In this way one-third of the values (ten in every half-degree) have been tested for gross errors.

In addition, I have differenced the values at 0(1′)1°10′ and 89°(1′)90°; moreover, I have checked the values at 0(1′)10′ and 89°50′(1′)90°, the former by comparison with the little table on p. 496 of INNES 2, and the latter by simple calculation therefrom. This supplementary work revealed no gross error in the regions at the beginning and end where differencing at interval 6′ or 5′ fails as a check.

The values at 0(6′)90° are entirely free from gross error. I have compared the values at 0(1°)90° with logarithms, to 10 or more D, of the values of q given in SPENCELEY 1. This revealed only two rounding errors, both very fine; at 33°, *for* 819, *read* 820 (rounding 819507), and at 42°, *for* 332, *read* 333 (rounding 332520).

Differencing at interval 5′ does reveal three gross errors and one rather large last-figure error:

θ	D	*For*	*Read*
4°50′	8	4	9
17 20	4	0	1
46 25	5	0	1
65 5	5	4	5

In confirming the corrections at 46°25′ and 65°5′ by differencing at interval 1′, it was found that in each case a neighboring value is affected by gross error: θ = 46°26′, 5D, *for* 6, *read* 7; θ = 65°4′, 5D, *for* 3, *read* 4.

On p. 14 the right-hand argument 11 is erroneously printed as 12.

In general the table is evidently accurate, but its 5400 values may perhaps contain 15 or 20 gross errors. Differencing of the whole table at interval 1′ is highly desirable.

MILNE-THOMSON 1.

(α) q, k^2 = [0(.01)1;10D].

Differencing suffices to show freedom from gross error up to at least k^2 = .88. For k^2 = .88(.01).99 I have deduced 10D natural values from the 10D table of log q in HAYASHI 1; in all cases I found agreement within one unit of the tenth decimal. Thus all the function values are free from gross error. Moreover, Milne-Thomson's differences accurately correspond to his function values.

MILNE-THOMSON 2.

(α) K, E, k^2 = [0(.01)1;9D].

"About 70 errors (in 200 values) of a unit each, one of two units, and none greater" (COMRIE, *Math. Rev.*, v. 7, p. 485). The error of two units is in K', at $m = k^2 = .01$, where *for final* 5, *read* 3. Comparison with the 10D values in HAYASHI 1, 3 confirms that in all other cases Milne-Thomson's values are within 1½ units of the truth, the greatest remaining discrepancy between the two values being one of 15 units of the tenth decimal, in K' at m = .37, where Milne-Thomson ends in 81, HAYASHI ends in 795, and I find 79528. I have not checked the differences, nor the table of $1/\vartheta_3^2 = \pi/2K$.

MILNE-THOMSON 3.

(α) K, E, k^2 = [0(.01)1;7D].

Free from error. I have read against both HAYASHI 1, 3 and Milne-Thomson 2.

(β) q, k^2 = [0(.01)1;8D].

Free from error, assuming the table correct in a few cases in which the 10D table in Milne-Thomson 1 ends in 50.

MOORE 1.

(α) 2E, k' = [0(.01)1;4D].

Since FMR *Index* was published, I have seen MOORE's table in the British Museum. It gives 2E (not 4E, as erroneously stated on p. 317 of FMR *Index*). The last two decimals are often false, and the table is of no interest apart from its early date.

NAGAOKA & SAKURAI 1.

(α) K, E, k^2 = [0(.001)1;6D].

"In comparing the first 250 values of K [with the 7D values of SAMOĬLOVA-ĨAKHONTOVA] only slight last figure differences were found in about a score of cases" (ARCHIBALD, *Scripta Math.*, v. 3, 1935, p. 365).

PLANA 1.

(α) log (1/q), θ = [0(0°.1)45°(1°)90°;10D].

The present writer was responsible for printing Plana 1848 in bold type on p. 321 of FMR *Index*. Further experience has unfortunately shown the implied recommendation to be quite unwarranted. Differencing immediately proved the table to be scandalously inaccurate. While it would not be difficult to recompute the whole 10D table (for instance, mainly by taking twelve-figure antilogarithms of VERHULST's values, corrected where necessary), I have lacked time, and shall confine myself to stating the numerous corrections necessary to

θ	PLANA (8D)	MEISSEL
1°.2	4.56207 605	4.56197 100
4 .2	3.47347 021	3.47347 740
5 .5	3.23897 197	3.23897 195
9 .4	2.77213 727	2.77215 327
10 .1	2.70946 393	2.70946 258
13 .4	2.46216 418	2.46216 318
13 .6	2.44917 384	2.44917 385
14 .1	2.41750 352	2.41750 252
15 .2	2.35152 745	2.35152 845
15 .7	2.32306 816	2.32306 793
15 .8	2.31748 196	2.31748 197
16 .0	2.30641 298	2.30641 257
16 .1	2.30092 808	2.30092 827
16 .3	2.29006 836	2.29005 836
16 .9	2.25789 743	2.25820 939
18 .0	2.20256 652	2.20256 662
18 .3	2.18794 474	2.18796 174
18 .4	2.18314 464	2.18314 474
18 .5	2.17835 396	2.17835 290
18 .8	2.16412 567	2.16412 568
20 .0	2.10931 747	2.10931 745
20 .3	2.09610 811	2.09610 832
20 .6	2.08308 433	2.08308 444
22 .0	2.02460 039	2.02460 036
22 .7	1.99667 024	1.99667 036
26 .2	1.86812 129	1.86812 124
26 .4	1.86126 771	1.86126 797
26 .5	1.85785 945	1.85785 926
26 .9	1.84434 183	1.84434 182
27 .0	1.84099 060	1.84099 133
27 .4	1.82769 382	1.82770 224
27 .5	1.82440 592	1.82440 774
27 .6	1.82112 718	1.82112 418
27 .7	1.81784 242	1.81785 147
28 .1	1.80486 789	1.80486 753
28 .2	1.80091 548	1.80164 786
28 .7	1.78520 349	1.78570 349
29 .4	1.76379 582	1.76379 882
29 .7	1.75455 482	1.75455 478
30 .2	1.73933 275	1.73933 272
30 .9	1.71839 436	1.71839 602
34 .9	1.60617 525	1.60619 744
35°.4	1.59296 681	1.59296 451
35 .6	1.58771 573	1.58771 568
40 .1	1.47743 564	1.47563 204
40 .9	1.45678 520	1.45678 185
41 .1	1.45211 467	1.45211 450
41 .6	1.44052 385	1.44052 284
42 .0	1.43132 888	1.43132 667*
42 .1	1.42900 828	1.42903 813
42 .2	1.42675 335	1.42675 374
42 .3	1.42417 316	1.42447 347
42 .4	1.42219 736	1.42219 730
42 .5	1.41992 512	1.41992 520
42 .6	1.41765 690	1.41765 715
42 .7	1.41539 320	1.41539 312
42 .8	1.41313 315	1.41313 309
42 .9	1.41087 671	1.41087 704
43 .7	1.39296 873	1.39296 884
44 .0	1.38631 594	1.38631 596
44 .2	1.38195 353	1.38189 922
44 .8	1.36873 522	1.36873 581
45 .0	1.36438 892	1.36437 635
46	1.34278 395	1.34278 396
48	1.30055 663	1.30055 763
49	1.27988 537	1.27988 450
50	1.25949 994	1.25947 994
53	1.19970 369	1.19970 347
54	1.18020 131	1.18020 158
55	1.16088 275	1.16088 475
60	1.06653 416	1.06653 431
63	1.01117 242	1.01115 242
86	0.52945 956	0.52945 958
88	0.45221 546	0.45202 155

Obvious errors in integral part only:

	For	*Read*
29°.9	7	1
64	1	0
65	1	0
66	1	0
67	1	0

For argument 86.6 read 36.6.

* At 42°.0, Meissel would wrongly give 668.

turn the table into a reasonably good 8D table. I have compared Plana's 10D values with MEISSEL's 8D values, or rather with their complements, since Plana tabulates log $(1/q)$ and MEISSEL log q. MEISSEL's values at arguments which are multiples of 6′ are entirely free from gross error and the eighth decimal is generally very accurate. On p. 277 are listed the 79 cases (out of 495 values!) in which Plana's value differs by more than 100 units (in fact, 122 or more) of the tenth decimal from MEISSEL's 8D value. Plana's values are given rounded to 8D. These are all gross errors in Plana. There are also more than a score of cases, not listed, in which Plana's value differs from MEISSEL's by between 51 and 100 units of the tenth decimal. Some of these will be due to errors of rounding in MEISSEL, but many will be gross errors in Plana, and there will probably be other gross errors in Plana not affecting the eighth decimal. The list on p. 277 aims partly at salvaging something from the chaos of Plana's table, which is fairly accessible, and partly at making more widely available to scientists at least the gist of MEISSEL's rare work.

POTIN 1.

(α) log q, θ = [0(5′)90°;5D].

Contains all the errors given under BERTRAND 1 (α) and none of those given under LÉVY 1 (γ). I have not read the table against BERTRAND to see whether Potin has introduced any copying errors.

(β) log ϑ_3 [called log $\theta_2(0)$], θ = [0(30′)90°;9D].

The function tabulated is $\frac{1}{2}\log(2K/\pi)$. Except that the integral part at 67°30′ is corrected, the table is faithfully copied from BERTRAND, and has the same errors as those already given in BERTRAND 1 (β).

(γ) $F(\theta, \phi)$, $E(\theta, \phi)$, θ = 0(1°)90°, ϕ = [0(1°)90°;9–10D].

See under LEGENDRE.

ROSENBACH, WHITMAN & MOSKOVITZ 1.

I have not examined the tables for errors. On two errors, see ARCHIBALD, *MTAC*, v. 2, p. 217, 1947.

RUNKLE 1.

Not too bad up to α = .5, rather unreliable from .5 to .7, and so unreliable as to be almost useless between .7 and .75.

SAMOĬLOVA-ŤAKHONTOVA 1.

I have not checked the extensive double-entry tables.

(α) K, E, k^2 = [0(.001)1;7D].

The values are based on HAYASHI 1, 3, which give at least three further decimals over the whole range. I have examined the tables only for those arguments at which HAYASHI has errors, and for two arguments at which Samoĭlova-Ťakhontova herself points out errors in the errata slip. The first and fourth of the following errors are due to errors in the source; the second and third are from the errata slip.

p.	k^2	Function	D	*For*	*Read*
90	.201	E	4	5	6
92	.310	K	2	2	1
94	.439	E	1–7	387428	3807428
105	.999	K	7	5	6

ARCHIBALD (*Scripta Math.*, v. 3, 1935, p. 365) has checked that the values of K for k^2 = 0(.01)1 are in agreement with the 9D values of MILNE-THOMSON 2.

(β) q, k^2 = [0(.001)1;8D].

This table falls for purposes of checking into two parts.

For k^2 = 0(.001).5, the function values were reprinted from HAYASHI 1, first differences being added. I have confirmed that the function values were correctly copied, the major

errors at $k^2 = .395$ and .482 being corrected; at $k^2 = .279$, the last-figure correction (*for final* 7, *read* 8) also given by HAYASHI in 1932 has, however, not been made. See HAYASHI 1 (γ). The function values are thus free from gross error. At $k^2 = .403$, the last digit is almost illegible in my copy, but the differences show that 5 is intended.

For $k^2 = .501(.001)1$, the table is original. Differencing shows freedom from gross error up to at least $k^2 = .988$, and I have recomputed the values at $k^2 = .986(.001).999$ to 10D, finding only two rounding errors. Comparison of every tenth value with the ten-decimal table at interval .01 in MILNE-THOMSON 1 also revealed two rounding errors, as well as two cases (at .58 and .79) in which MILNE-THOMSON 1 ends in 50, and in both of which recomputation showed HAYASHI to be wrongly rounded. Recomputation of a small stretch with rough differences round $k^2 = .977$ discovered three more rounding errors. The various rounding errors referred to are:

k^2	.580	.670	.790	.975	.977	.979	.980	.992	.997
for	8	0	8	7	8	4	8	6	2
read	9	1	7	8	9	5	7	5	3

Further rounding errors doubtless exist, but in general the final digit is evidently accurate.

The differences are printed in small and occasionally defective type, a few digits being barely legible, but, as far as I can see, the differences in the whole table of q accurately correspond to the printed function values.

SCHLÖMILCH 1.

(α) $E, k' = [0(.01)1;5D]$.

Comparison with the accurate 11D table of SCHMIDT 1 shows that Schlömilch's values are all correct, if his last-digit underlinings are disregarded. The last digits should be underlined (to denote tabular values in excess of the true values) at .07 and .60.

SCHMIDT 1.

I have not examined the extensive double-entry tables.

(α) $E, k' = [0(.01)1;11D]$.

These are quadrants of ellipses with semi-axes 1 and k'. The table is by far the most important of its kind, and it is unfortunate that it has not been more widely known, as it is thoroughly reliable. Differencing, supplemented by recomputation at $k' = 0(.01).10$, shows that there are no gross errors. It should be noted that, in all cases where my information goes beyond the eleventh decimal, Schmidt's value has been obtained by rounding downwards, that is, he has not raised the digit in the eleventh place, even when the following digit exceeds 4. Presumably this holds throughout the table.

SPENCELEY 1.

I have not examined the double-entry tables. In the page headings are given three transcendental functions, namely K, E, q. The quantities $D(90)$ and $1/D(90)$ also given in the headings are not transcendental, being equal to $1/\sqrt{k'}$ and $\sqrt{k'}$ respectively, and thus merely algebraic functions of the modulus. I have examined the values of K, E, q as given (each twice, without any disagreements) on p. 2–181; the values are given another twice in the second half of the table, which I have not examined.

(α) $K, \theta = [0(1°)90°;15D]$.

I have differenced the values for 0(1°)81°; at the end this requires twenty-second differences. The process proves the Spenceley values to be free from gross error for $\theta = 0(1°)69°$; beyond 69°, the table cannot be checked to the full 15 decimals by differencing. For $\theta = 70°(1°)89°$, 13D values of my own agree with the Spenceley values to that number of decimals, so that the whole table is certainly accurate enough for all practical purposes.

(β) $E, \theta = [0(1°)90°;15D]$.

I have differenced the values for $\theta = 0(1°)83°$, and so proved freedom from gross error up to at least 73°. For 70°(1°)89°, my own calculations check the Spenceley values to at least 13D. Thus the whole table is again of great accuracy.

(γ) q, $\theta = [0(1°)90°;16S]$.

I first differenced the values for $\theta = 0(1°)71°$ to 16D, rounding to this number for $\theta = 0(1°)63°$, where further decimals are given. This proved freedom from gross error to 16D up to at least 63°. The process could be pushed a few degrees higher. I have, so far, checked the values above 63° only to the extent that I have derived 10D logarithms of all the Spenceley values of q, and these agree with 9D logarithms which I had earlier deduced from the K and K' of Legendre 7, using $\log q = -CK'/K$, where $C = \pi \log e$. This check will suffice for most purposes.

Later, out of curiosity, I differenced the Spenceley values of q for $\theta = 0(1°)22°$ to 18D, and to my surprise found incontestable evidence of errors in the eighteenth decimal. I have, so far, recomputed q to 20D for $\theta = 1°(1°)12°$. These values difference well, and give the following corrections, the first digit mentioned being in the sixteenth decimal in each case:

θ	*For*	*Read*	θ	*For*	*Read*
3°	4727	4708	9°	192	198
5	8574	8571	10	284	286
6	8341	8369	11	765	770

At the other six arguments I agree completely with the Spenceleys. The largest error so revealed is 6 units in the eighteenth decimal at 9°; the largest error expressed in units of the last place kept by the Spenceleys is 28 final units at 6°.

I have not yet differenced the values up to 63° to 17D. Anyone not content with 16D in this range should difference or recalculate. The errors which I have found are unlikely to affect the main Spenceley tables.

Verhulst 1.

Verhulst says (p. 252) that the table of log log $(1/q)$ was calculated for him by Loxhay, a young Belgian mathematician, from Legendre's tables. The formula is

$$\log \log (1/q) = \log K' - \log K + 0.13493\ 41839\ 94670\ 6.$$

I shall deal with the table in two parts.

(α) log log $(1/q)$, $\theta = [0(0°.1)15°;14D]$.

After differencing, I first recalculated all values, correcting log K as in Legendre 8 (α) and log K' as in Legendre 8 (γ); in the latter case Legendre's table is, of course, now regarded as a table of log K' for $\theta = 0(0°.1)15°$ rather than as a table of log K for $\theta = 75°(0°.1)90°$. As log K and log K' so obtained are still liable in most cases to considerable last-figure errors, my results are in general very uncertain in the 14th decimal, but are sufficient to correct Verhulst into a table to 13 working decimals.

There are two gross errors due presumably to slips by Loxhay: $\theta = 1°.2$, 5D, *for* 6, *read* 5; $\theta = 4°.2$, 6–7D, *for* 35, *read* 44. These are the only errors due to Loxhay.

θ	*For*	*Read*	θ	*For*	*Read*
0°.7	795573	805573	5°.4	86983	87050
1 .1	35632	35631	5 .5	12965	13165
1 .3	95056	95456	5 .6	83793	83994
2 .9	56794	56799	5 .7	30859	31059
3 .1	79255	79397	6 .3	81232	81131
3 .2	60913	57874	6 .8	71013	71048
4 .3	74463	74472	7 .0	91125	91115
4 .7	25150	25183	9 .7	27826	27723
4 .8	57083	57282	9 .9	92341	92336
4 .9	33199	33397	10 .5	99630	99632
5 .0	83774	83960	10 .6	51339	51341
5 .1	83060	83260	10 .9	13635	13660
5 .2	85112	85311	11 .4	33851	33861
5 .3	11686	11886			

The errors at the 27 arguments corresponding to those listed under Legendre 8 (α, γ) are given at the bottom of p. 280, the last digit being always in the fourteenth place. The list probably omits no error of more than ten final units in Verhulst. The corrections are slightly provisional, but are liable to errors of only a unit or so in the last place, as I later applied a few small corrections to log K not listed under Legendre 8 (α), and kept extra decimals in the calculations when I knew them. At 1°.1 Legendre's errors in log K and log K' nearly cancel, so that Verhulst is nearly correct.

(β) log log $(1/q)$, θ = [15°(0°.1)45°;12D].

There are exactly four gross errors:

θ	D	*For*	*Read*
15°.5	11	4	3
21 .0	9–10	22	16
30 .9	8	6	5
37 .7	12	9	0

The correction to the last figure at 37°.7 is approximate, being that which brings Verhulst's value into agreement with Legendre's log K and log K', and also rectifies the differences. When the above corrections are made, the table differences very well, especially when it is remembered that it was formed by subtracting Legendre's log K from his log K' (and adding a constant, which, however, will not affect the differences); even if Legendre had no rounding errors, Verhulst's values could frequently be wrong by one final unit. A similar remark applies to the 14D portion considered in (α) above. Verhulst's values are unaffected by the two gross errors (one obvious) at 24°.9 and 30°.9 in Legendre 5, listed under Legendre 8 (β), his error at 30°.9 being quite different; thus he probably used Legendre 3.

Wayne 1.

(α) E, k' = [0(.01)1;3D].

Comparison with Schlömilch 1 and Schmidt 1 shows two slight rounding errors; at k' = .20, *for* 1.050, *read* 1.051, and at k' = .81, *for* 1.426, *read* 1.425.

The Memory Tube and Its Application to Electronic Computation

In a paper presented on 10 June 1947 at the IRE Electron Tube Conference, the author described a novel electronic device which is called the Memory Tube.[1] The Memory Tube has the ability to record electrical signals in the form of a charge pattern deposited by an electron beam on a dielectric target, to store this information for any desired period of time, and to reproduce the stored information in the form of electrical signals as many times as desired, without erasing the recorded information. The recorded charge pattern or any part of it can be erased whenever desired. These characteristics of the Memory Tube make it useful for many applications, including electronic computation. The purpose of this paper is to review the basic principles of operation of the Memory Tube and to describe a special form suitable for use as a memory device in an electronic computer.

The Memory Tube resembles a conventional cathode-ray tube. At one end of an elongated envelope are placed one or more electron guns. At the other end is placed a dielectric target, which ordinarily consists of a glass plate coated with a suitable material, such as phosphor. In front of the target a fine-mesh metal screen is placed at a distance of a few thousandths of an inch from the target. The screen serves to control the electric field at the

target surface and to collect secondary electrons emitted from the target. The charge pattern is deposited by a well-focused electron beam which "writes" on the particular target element to which it is directed. An auxiliary electron gun producing a diffuse electron stream continuously bombards the whole area of the target. The velocity of electrons in this "holding" beam is so adjusted that electrons arriving at positive areas tend to maintain those areas at collector potential, while electrons arriving at negative areas drive these areas to holding gun-cathode potential. This result is achieved because of the secondary emission characteristics of the target,—

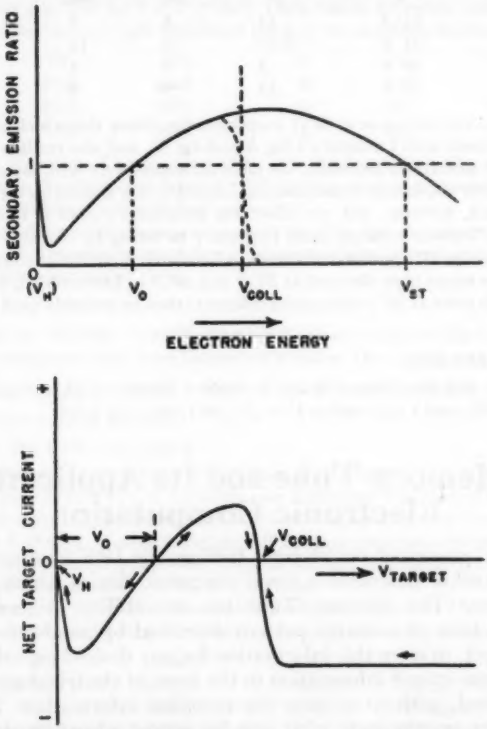

FIG. 1. SECONDARY EMISSION CHARACTERISTICS OF THE MEMORY TUBE TARGET.

namely, that the net current flowing to the target element is negative when the electron velocity is below a critical value V_0 corresponding to unity ratio of secondary emission, and is positive when the velocity exceeds this critical value (see Fig. 1). Since the velocity of electrons arriving at a target element is determined by the potential of that element, the positive areas are bombarded with high-velocity electrons, and the negative areas with low-velocity electrons, even though electrons originate from the same source.

In order to change the element potential from one level to the other, the "writing" beam must have high current density to overcome the stabilizing

effects of the holding beam. When the writing beam velocity is high, a positive charge will be deposited on the target; when its velocity is low, a negative charge will be deposited. Thus, the intense writing beam can change the potential of any element (to which it is deflected) either in the positive or in the negative direction, irrespective of the previous condition. Since the erasure is done by the writing beam itself during the writing process, the stored charge pattern can be modified in accordance with the new information without requiring a special erasure process.

In order to reproduce stored information (the process usually referred to as "reading"), the target surface is "probed" with a well-focused electron beam. The net current to the collector, attributable to the "reading" beam, depends upon the transparency of the collector, the number of secondaries generated by the electrons at the target, and the direction and magnitude of the electric field at the two surfaces of the collector electrode. When the reading beam is directed towards a negative area of the target, the secondary electrons from the target pass through the collector screen with a velocity equal to the difference of potential between the negative areas and the collector. In addition, there is a strong suppressing field near the collector. When the reading beam is directed towards positive target areas, the secondaries from the target are not accelerated towards the collector, as is the case near the negative areas, but are in part turned back to the target surface. This occurs under the equilibrium condition since the positive areas of the target are higher in potential than the collector by a few volts because of initial velocities of secondary electrons. The differences in velocities of returning electrons and the differences in electric fields near the collector for the positive and negative areas of the target result in a corresponding difference in the collector current. Thus, the magnitude of the collector current depends upon the polarity of the charge on the target surface, and the probing by the reading beam indicates whether a positive or a negative charge has been stored at the element being probed.

From the above description it can be seen that the Memory Tube can store information in the form of charges on the surface of the target, and the charge distribution can be ascertained by probing with the reading beam. These characteristics make the Memory Tube suitable for use in electronic computers.

The function which the computer memory device is required to perform is to record, to store, and to read out numbers whenever desired, and to record modifications of these numbers which result from mathematical operations upon them. By the use of the binary system any number can be represented by a pattern of 1's and 0's, which can be represented on the Memory Tube target as a pattern of positively and negatively charged "dots." If to each digit is assigned a definite elementary area of the target, then the process of recording a digit consists in deflecting the writing beam to the particular spot and depositing either a positive or a negative charge. The reading process consists in ascertaining with the reading beam the polarity of the charge on the particular element probed.

In order to record and read as many digits as possible, the resolution of the Memory Tube—that is, the number of discrete areas on the target that can be independently charged to either positive or negative potential—should be as high as possible. This requires that the writing and the reading

beams be well-focused, and that the target be of such nature that under the action of the holding beam no diffusion of charges over the target areas can take place. In addition, the accuracy in positioning of the writing and reading beams should correspond with the required resolving power. This last requirement makes it very desirable to use the same beam for writing and for reading. This is possible since in the computer tube the writing and reading processes are not simultaneous.

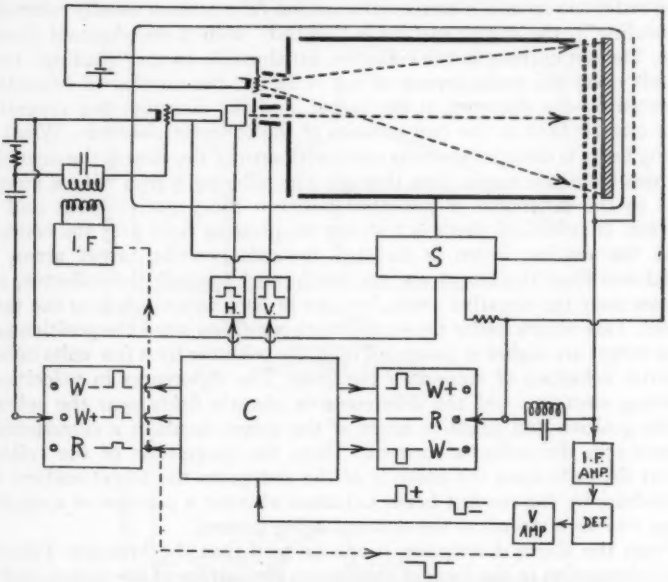

FIG. 2. SCHEMATIC DIAGRAM OF THE COMPUTER TYPE MEMORY TUBE AND THE ASSOCIATED CIRCUITS.

A schematic diagram of a Memory Tube intended for computer use is shown in Fig. 2. The write-read gun of the conventional cathode-ray tube type produces a beam focused on a phosphor-coated target. A special holding gun generates a diffuse electron beam, which illuminates the target over its entire area. Two metal woven-mesh screens are placed in front of the target. The screen closest to the target controls the field at the target surface and serves to collect secondary electrons from the target. This screen is ordinarily used as the output electrode. A second screen, farther away from the target, serves to maintain constant the distribution of electric fields along the path of the electron beam irrespective of variable potentials which can be applied to the collector screen. The modulation of collector potential is necessary in order to vary the terminal velocity of electrons for writing either positively or negatively and for reading by means of the same electron beam. The constancy of electric fields along the beam is dictated by the requirement of

accurate positioning of the write-read beam, irrespective of its terminal velocity.

The diagram of Fig. 2 shows schematically the control pulses originating in the computer represented on the diagram by block C. The different control pulses are necessary in order for the tube to perform the operations of writing, reading, and holding. For writing negatively, a strong positive voltage pulse is applied to the grid of the write-read gun. Simultaneously, a negative voltage pulse is applied to the collector screen. This causes an intense beam of slow electrons to arrive at the target and to deposit a negative charge. In order to deposit a positive charge, the grid of the write-read gun is again pulsed simultaneously with the pulse on the collector, which this time is pulsed in the positive direction. The resulting high terminal velocity of the electrons arriving at the target (which also changes its potential during the pulse because of capacitive coupling between the collector and the target) produces copious secondary emission from the bombarded element of the target and changes it to a positive potential.

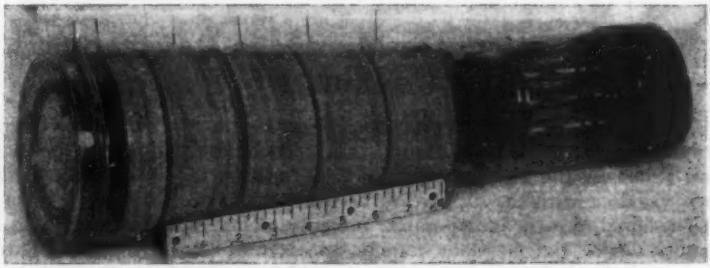

FIG. 3. A PHOTOGRAPH OF A DEVELOPMENTAL MODEL OF THE MEMORY TUBE.

While the terminal velocity of electrons in the writing beam is varied in order to achieve either positive or negative writing, the terminal velocity of electrons in the holding beam should be maintained at a constant value for maximum stability of the charge pattern. This is accomplished by applying to the cathode of the holding gun the same pulse voltages which are applied to the collector electrode, as is shown schematically in Fig. 2.

In order to minimize spurious signals caused by modulating pulses and switching operations during the read-out process, the reading beam is modulated at a frequency of several megacycles. As a result, the output current from the collector screen is also modulated at the same frequency and can be amplified in a selective amplifier (similar to the usual intermediate-frequency amplifiers) which effectively rejects low-frequency components caused by switching. As is shown in the diagram of Fig. 2, a high-frequency voltage is applied to the grid of the write-read gun in addition to a small positive voltage pulse during the read-out process while the collector is maintained at its normal potential. The output signal consisting of an i–f modulated voltage pulse appearing across the circuit tuned to i–f is amplified in the i–f amplifier, then detected and further amplified in a video-amplifier. Since the probing of positive and negative elements of the target

results in output pulses of different amplitudes, it is possible, by mixing a constant-amplitude negative pulse in the video amplifier, to obtain as a final result a positive pulse for a positive element and a negative pulse for a negative element of the target.

A developmental model of the Memory Tube intended for computer use is shown in Fig. 3. The general shape of the tube, the relative position of the guns and screen-and-target structure can be seen. Since the computer Memory Tube is still in the development stage, it is expected that many improvements in the design of both the tube and the associated circuits will be made before a completely satisfactory performance is achieved. It is hoped that the above brief account of the present state of development will stimulate workers in this field to explore further the potentialities of the memory device which has been made available to them.

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Naval Research Laboratory
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[1] A. V. HAEFF, "A memory tube," *Electronics*, Sept. 1947, p. 80–83.

The Logical Design of the Raytheon Computer

The design to be described in what follows is that of a high-speed digital computer for general scientific work. It is intended to meet general specifications issued by the National Bureau of Standards, for whom most of the work was done with the support of the Office of Naval Research.

The final design incorporates mercury delay lines as an internal memory device and magnetic tape as an external storage medium. These features of the design, together with the NBS specifications, dictated most of the machine parameters.

The basic logical units of the computer are: (1) internal memory, (2) external memory, (3) arithmetic unit, (4) central control, which constitute what might be called the high speed part of the machine, and two auxiliary units: (5) problem preparation unit, (6) printer.

The Memory and the Representation of Numbers and Orders.—The computer handles several different kinds of numbers. In standard or "normal" operation, numbers are stored to a precision of 35 binary digits and with a fixed binary point. The normal precision of the basic arithmetic operation is also 35 digits. Numbers of 70 binary digits—i.e. numbers of double-precision—can be stored and manipulated as pairs of standard numbers. Floating-point numbers can also be handled as pairs of standard numbers, one of these being a number between $\frac{1}{4}$ and $\frac{1}{2}$, and the other being an exponent on 2. Since decimal-to-binary and binary-to-decimal conversions are performed in the arithmetic unit, instead of in auxiliary equipment, decimal numbers must also be stored in the internal memory. These are stored in the binary-coded decimal notation to a normal precision of 8 decimal digits. Decimal numbers of double-precision can be handled as two normal numbers.

Not only numbers but coded instructions (which will henceforth be called orders) as well are stored in the internal memory. An order contains enough information to specify a complete arithmetic operation, including

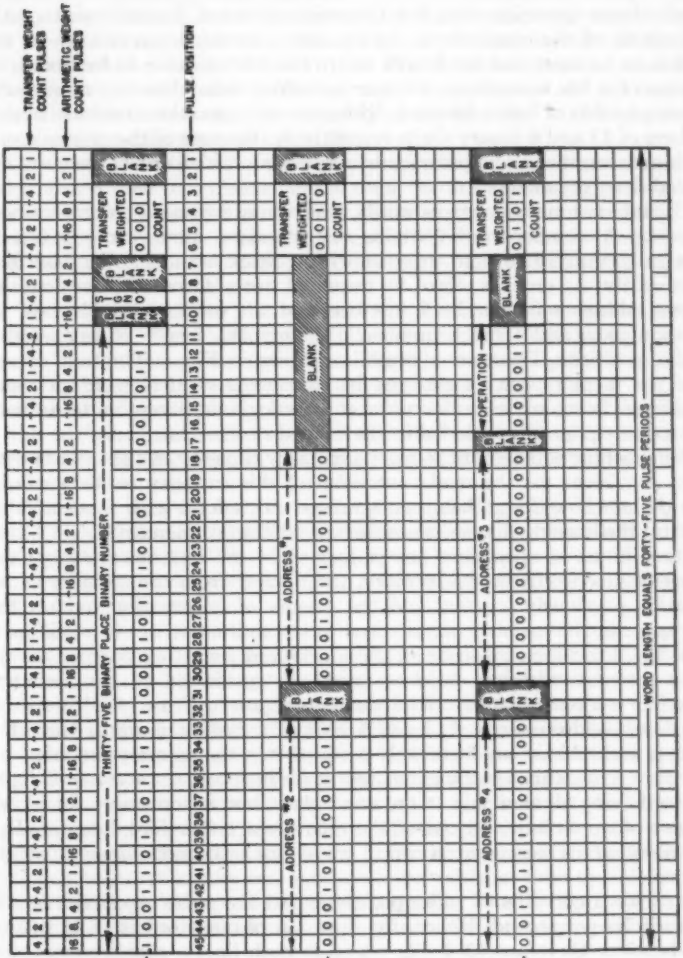

FIG. 4. ALLOCATION OF INFORMATION IN NUMBER- AND ORDER-WORDS.

all the required selections in the memory. Most orders consist of four address codes and one operation code. For these orders, two of the addresses specify the location of the operands in the memory, the third specifies where the result is to be sent, and the fourth where the next order is to be found. In the case of a few operations, the corresponding order does not require the full complement of four addresses. Addresses and operation codes are binary numbers of 13 and 6 binary digits respectively. Because of the presence of a fourth address, there is no preassigned sequence in which orders are normally selected from the memory.

The internal memory stores words, which may represent either numbers or orders. A number-word contains one standard number of 35 binary digits and an algebraic sign, together with a check number to be described subsequently. An order is stored by means of two order-words, one containing two addresses and a check number, and the other two addresses, an operation code, and a check number. The complete length of any word is 45 pulse positions. These are allocated as shown in Fig. 4. The total capacity of the memory can be changed in increments of 1024 words without any changes being made in other units of the calculator, except in the amount of selection equipment included in the central control unit.

The machine will be built to have a memory capacity of 1024 words with design provisions for the ready extension of this capacity to 4096 words. Since the machine design has been arranged to conform with this extended capacity, the ensuing discussion assumes a 4096-word memory.

Three kinds of operations are performed in the internal memory: the writing of new words in the memory, the reading from the memory of information previously written, and the erasure of information so that the same space in the memory can be used to store different words at different times. The logical organization of the machine is such that erasure always occurs automatically just before a word is transferred to the memory. In the four-address order described above (except in the case of a few special operations), addresses nos. 1, 2 and 4 control transfers of words from the memory, and address no. 3 controls a transfer to the memory preceded by an erasure. Words are serially circulated in the memory, the successive digits of a word being all available at the same place, but at successive times. All transfers of words into and out of the memory are therefore done serially; the pulses which represent a word travel in succession along a single "channel."

Each word contains 45 pulse positions, and the pulse repetition rate is 4 megacycles per second. Hence, the duration of one word is 11.25 microseconds; since there are 32 words stored in each line, a complete memory cycle has a duration of 360 microseconds. The time required to select a word in the memory, and to transfer it to the arithmetic unit varies from 2 to 33 word durations.

The external memory is used to supplement the internal memory. Four magnetic-tape units are provided in the design, each capable of storing as many as 10^6 words. Thus, nearly a half million different words can be automatically processed by the machine without any human intervention being required. The different words stored in the external memory have vastly different accessibility times. Any tape can be scanned at the rate of about 500 words per second; hence, in the most unfavorable case it may take as

long as 200 seconds to read a given word out of the external memory. The most efficient use of the external memory occurs when the recorded data can be used in the sequence in which they appear on the tape. If the data must be used in an order different from that in which they are recorded, a mass rearrangement prior to reading might be performed. If this is impossible, the effective computational speed of the machine would conceivably be reduced by a factor as great as 10^4.

Not only does the external memory supplement the internal memory, but it also is used to introduce initial data—both numbers and orders—into the calculator. These initial data are recorded manually on magnetic tape by means of the problem preparation unit. When the tape so prepared has been manually placed on a magnetic-tape unit, the recorded data are automatically available to the machine.

The external memory also serves as a link between the machine and the output printers. Results obtained by the computer are recorded by means of one of the four magnetic-tape units. The tape containing the data is then manually placed on one of the two automatic printing units.

Since the decimal-binary and binary-decimal conversions are performed in the arithmetic unit, original input and final output numbers handled by the external memory are in the binary-coded decimal notation; all other numbers and all orders are in the binary notation.

Words are recorded on the tape in groups of 32, each group having associated with it a tag for identification purposes. Reading from the tape is also done in blocks of 32. Words are transferred directly between the external memory and the arithmetic unit only and not between the external memory and the internal memory.

Associated with each tape unit are two 32-word delay lines, or reservoirs, which are used as buffers between the main electronic part of the machine and the tape. In recording, the words to be recorded are all sent to one reservoir until it is filled; at this point they are sent to the second reservoir until that in turn is filled, after which they are once again transmitted to the first. As soon as a reservoir is filled, it is emptied automatically onto the tape. Thus, computation and recording can proceed simultaneously. As the machine computes, the results are sent to one reservoir; at the same time, the contents of the other reservoir (computed previously by the machine) are being recorded on the tape. If the computer generates words faster than they can be recorded, the computation is halted from time to time to allow the recording to keep pace; when the opposite condition prevails, the recording is halted periodically. In reading from the tape, the use of the reservoirs is essentially the same as that described above for recording, except that the flow of words is in the opposite direction—from tape to reservoir to arithmetic unit.

Facilities are provided for random hunting on each tape. The process of hunting consists in looking for a block with a given tag, and then in reading that block, or under certain circumstances that block and the next, into the reservoirs. Words are recorded on the tape in four parallel channels.

All storage of words in both the internal and external memories, and all transfers of words are checked by means of a weighted count, i.e., a weighted sum of the digits of the word (*including the algebraic sign* if the word is a number-word). This weighted count is stored with the number or half-order

as part of the word and is called the transfer weighted count. The weights chosen for the sum are the numbers 1, 2, 4, 1, 2, 4, 1, 2, 4, etc., which are assigned to the successive digital positions from right to left. This weighted sum is computed modulo 16 and is then modified by the addition of 1; thus, a number and its weighted count cannot both be zero simultaneously. With this modification, a null word is not a valid word, and the complete failure of a gate or other device controlling the entire transfer channel will be detected. Since the sum is computed modulo 16, only four digital positions instead of seven are required to represent the weighted count.

Arithmetic Unit.—The arithmetic unit is the only unit in the machine capable of generating new numbers or new orders. All of the other machine units have a passive role in the actual computation process, being used merely to introduce operands into the arithmetic unit and to call for the proper arithmetic processes.

Unlike the internal and external memories, the arithmetic unit uses a parallel representation of words, the various digits of the word being stored in separate devices (flip-flops). This permits a considerably higher computing speed than could be obtained in a serial unit. Transfers of words between the memory and the arithmetic unit are accompanied by serial-to-parallel or parallel-to-serial conversions. Arithmetic operations can be performed on either numbers or orders. Built-in operations on numbers include, first, the basic arithmetic operations of addition, subtraction, multiplication, and division; second, transfers of numbers from one memory position to another with at most a change in the algebraic sign of the number; third, number shifts; and fourth, extractions of any preassigned set of digits from a number. Built-in operations on orders (strictly speaking on order-words, i.e., halves of orders) include addition, subtraction, and transfer.

The general way in which these operations are performed can be described in terms of the three registers, or number-storing devices, of which the arithmetic unit is composed. Denote these by the letters A, B, and C. Of the three registers, only B has provision for addition in the form of an accumulator. The number standing in A can be added to that in B, and the sum will appear in B. Numbers stored in registers A, B, and C will be denoted by X_A, X_B, and X_C.

In the process of addition involving standard numbers or order-words, the augend is first read from the memory to B, and the addend from the memory to A. Then the addition $X_A + X_B$ is performed, after which the sum can be read out of B. Negative numbers are stored in the memory as absolute values, with the algebraic sign appended, instead of their being stored as complements. Arithmetic subtraction, however, can only be performed by taking the complement of the subtrahend and adding it to the minuend. Hence, under certain circumstances the augend or the addend must be complemented before the addition takes place, and the sum may have to be complemented before it is sent to the memory. The complement that is used is the complement of $2 - 2^{-35}$; it is the original number subtracted from a number all of whose digits are unity. Thus, the complement is found merely by replacing all 0's by 1's and 1's by 0's. The use of this type of complement requires carry from the extreme left-hand column to the extreme right-hand column under some circumstances. The process of subtraction is similar to that of addition as described above.

In the multiplication of standard numbers, the multiplicand is first read to A and the multiplier to C. Each step of the multiplication then consists of the following: if the extreme right-hand digit of X_C is unity, X_A is added to X_B; if that digit is zero, null is added to X_B; then X_B (the partial product) is shifted one column to the right, as is also X_C. The digital columns of X_B which are shifted off the right end of the B register are read into the C register to take the place of the digits of X_C as they are shifted to the right. Thus, at the end of the multiplication, the complete 70-digit product stands in registers B and C. Either 35-digit section of this product, or both sections, can be utilized for subsequent calculation.

In the division of standard numbers, the dividend is transferred to register B, the divisor to register A, and the quotient is formed in register C. The complement of the number in A is first taken. Each step of the division then proceeds as follows: The complement of the divisor is added to X_B. If the sum is positive, unity is added into the right-hand digital position of register C. If the sum is negative, null is added into C, and the complement of the number in A is taken (restoring X_A to its original form). The divisor is then added into B. The complement of the number in A is recomputed. Finally, the remainder in B is shifted one column to the left, as is also the quotient in C. At the end of the division process, both the quotient and the remainder are available.

The operations of addition and subtraction give exact results, and hence require no round-off. If the low-order part of a product is discarded, however, the resulting half-product will generally be inexact. If no compensation were made for the discarded portion, products would be subjected to a mean bias of about one-half unit in the last place retained, i.e., a bias of $\frac{1}{2}\cdot 2^{-35}$. The root-mean-square error would be about $(1/\sqrt{3})\cdot 2^{-35}$. The occurrence of such a large bias is a particularly serious matter, since repetitive multiplication would lead to excessive resultant values of bias and r.m.s. error. The bias may be reduced by a factor of about 2^{-35}, and the r.m.s. error by a factor of $\frac{1}{2}$ if $\frac{1}{2}\cdot 2^{-35}$ is added to the complete 70-digit product before the right-hand 35 digits are discarded. The use of the above round-off process is optional, multiplication codes being provided for both a rounded and an unrounded product.

The operation of division may also produce an inexact result. If, after 35 digits of the quotient are obtained, the division process is terminated, there is usually a remainder. This is positive (considering only the absolute values of the divisor and the dividend). Merely neglecting the remainder would result in a bias and an r.m.s. error of $\frac{1}{2}\cdot 2^{-35}$ and $1/\sqrt{3}\cdot 2^{-35}$ respectively. It is not convenient to add $\frac{1}{2}\cdot 2^{-35}$ to the quotient to reduce the magnitude of these quantities, since register C, which stores the quotient, is not provided with facilities for addition. Hence, the alternative scheme of merely replacing the right-hand quotient digit by 1 is used. This greatly reduces the bias, but does not change the r.m.s. error.

So far, only arithmetic operations on standard numbers have been described, but double-precision and floating-point operations can be performed also. These are not built-in operations, but are compounded of standard operations which are in some cases slightly modified.

Consider the addition of two double-precision numbers. Each double-precision number is stored as two number-words, it being understood that

the number in one of the words is multiplied by 2^{-35}. In adding $X_1 + X_2 \cdot 2^{-35}$ and $Y_1 + Y_2 \cdot 2^{-35}$, two ordinary additions are performed. First X_2 and Y_2 are added, and the sum is left in the arithmetic unit, having been transferred from register B to C. There may, however, be a carry digit in the units column of $X_2 + Y_2$; this is transferred to the lowest order column of B. Then X_1 and Y_1 are added into register B. It is now necessary only to complete certain carries in order to obtain the double-precision sum. The carries in question occur whenever it is necessary to carry from the extreme left-hand digit of the 70-digit sum to the extreme right-hand digit. (As previously mentioned, such a carry is sometimes required because of the scheme of complementing used.) In order to complete these carries it is necessary first to interchange X_B and X_C, perform certain of the carries, and then interchange them again, performing the remainder. Double-precision subtraction is performed similarly.

In the multiplication of two double-precision numbers, $X_1 + X_2 \cdot 2^{-35}$ and $Y_1 + Y_2 \cdot 2^{-35}$, the products X_1Y_1, X_1Y_2, and X_2Y_1 are computed, the first being kept as a double-precision number. These products are then added together by means of double-precision addition.

In double-precision division, the reciprocal of the divisor is first obtained to 35 places by means of a standard division. It is then obtained to 70 places by means of an iteration formula. Note that double-precision operations must be used here. Finally, the double-precision reciprocal is multiplied by the double-precision dividend to obtain the result.

The floating-point operations comprise a second type of nonstandard arithmetic operation. These processes deal with numbers X of the form $X_0 \cdot 2^P$ where $\frac{1}{4} \leq X_0 < \frac{1}{2}$ and P is an integer. X is stored in two words, one containing X_0, and the other $P \cdot k$, where $k = 2^{-35}$. As illustrations of floating-point operations of the proposed calculator, addition and multiplication will be discussed.

To add the two numbers $X = X_0 \cdot 2^P$ and $Y = Y_0 \cdot 2^Q$ to obtain $Z = Z_0 \cdot 2^R$, where $\frac{1}{4} \leq Z_0 < \frac{1}{2}$, the machine must perform these processes: (1) Compare Pk and Qk, and compute $-|Pk - Qk|$. This is done by means of a normal branch. (2) If $Pk \geq Qk$, (a) use a controlled shift operation to obtain $Y_0' = Y_0 \cdot 2^{-|P-Q|}$. (b) Then add X_0 and Y_0' by means of a special addition, viz., addition (floating). At the end of this addition the quantities Z_0 and Sk have been computed, where $X_0 + Y_0' = Z_0 \cdot 2^S$ and $\frac{1}{4} \leq Z_0 < \frac{1}{2}$. (c) Compute $Rk = Pk + Sk$ by ordinary addition. (3) If $Pk < Qk$, (a) use a controlled-shift operation to obtain $X_0' = X_0 \cdot 2^{-|P-Q|}$. (b) Then add X_0' and Y_0 by means of addition (floating); at the end of this addition the sum stands in the form $Z_0 \cdot 2^S$, where $\frac{1}{4} \leq Z_0 < \frac{1}{2}$. (c) Compute $Rk = Qk + Sk$ by ordinary addition.

The floating-point multiplication of X and Y to obtain Z is performed as follows: (1) Use the multiplication (floating) operation to compute Z_0 and Sk, where $Z_0 \cdot 2^S = X_0 \cdot Y_0$ and $\frac{1}{4} \leq Z_0 < \frac{1}{2}$. (2) Compute $Tk = Pk + Qk$ by ordinary addition. (3) Compute $Rk = Sk + Tk$ by ordinary addition.

Several operations other than the four basic operations of arithmetic can be performed on numbers or orders. Transfers of numbers or orders involve the arithmetic unit only in an incidental manner. In the case of numbers, the algebraic sign can be changed during the transfer. Thus, if X is the number extracted from the memory, and Z is the number returned to the

memory, any one of four relationships can hold between X and Z: $Z = X$, $Z = |X|$, $Z = -|X|$, $Z = YX/|Y|$. In the last case, Y is a control number which is used to govern the sign of Z. (As an application of the fourth type of transfer consider the computation of sin X for $-A < X < A$ using only a table of sin X for $0 \leq X < A$. Compute sin $|X|$, and then sin $X =$ X sin $|X|/|X|$).

Three types of shift operations are provided: (1) Shift factor (normal)—The nonzero number X is given in standard form, viz., $2^{-35} \leq |X| < 1$. X_0 and $P \cdot 2^{-35}$ are obtained as two standard numbers such that $\frac{1}{4} \leq X_0 < \frac{1}{2}$, and P is an integer. (This is useful in computing logarithms, and in converting numbers from standard to floating-point form.) (2) Shift factor (square root).—The non-zero number X is given in standard form. X_0 and $P \cdot 2^{-35}$ are obtained as two standard numbers such that $\frac{1}{4} \leq X_0 < 1$, and P is an even integer. (This is useful in computing square roots.) (3) Controlled shift.—Two standard numbers X_0 and $P \cdot 2^{-35}$ are given. $X = X_0 \cdot 2^P$ is obtained as a standard number. This operation is the inverse of (1). (It is involved in computing square roots and in converting numbers from the floating-point to the standard form.)

The extraction of digits from a number X is accomplished by utilizing a preassigned control number Y containing a 1 in every digital position in which the extraction is to be performed. The result of extraction is then a number Z containing a 1 only in each column in which both X and Y contain a 1.

Operations on order-words include addition, subtraction, and transfer, all of which are performed on order-words exactly as on numbers; in fact, the arithmetic unit does not distinguish between order-words and number-words. Other arithmetic operations on order-words are not excluded, but would probably serve no useful purpose. It may not be quite obvious why one wishes to perform additions or subtractions involving orders. This type of operation is useful under two conditions. First, assume that it is not possible to specify one of the addresses in a particular order in advance, because the address depends upon a partial result obtained previously in the calculation. It is necessary, then, to be able to insert the address by addition as soon as it is known. Second, it is often advisable to use one order repeatedly for different operations which may require the insertion of different addresses. Using one order many times instead of many different orders conserves space in the internal memory which may be needed for other purposes. The coded instructions for a matrix multiplication, for example, embody but a single multiplication order. The addresses of the multiplication order are modified in accordance with the memory locations which contain the elements of the matrices. Using this scheme, the entire multiplication of a matrix of n rows and m columns by a matrix of m rows and p columns, requires only 16 orders, regardless of the size of the matrices.

The branch is one of the most useful of all operations. It permits the machine to select alternative computing routines. Normally the fourth address of each order specifies the location in the memory in which the next order is to be found. The branch (normal), however, selects for the address of the next order either the third or the fourth address, depending on the sign of a difference $X - Y$. If $X - Y \geq 0$, the third address is chosen, otherwise the fourth. An example of the utility of the branch is manifest in

the floating-point operations described above, where there are many alternative computing routines needed. A second kind of branch is available in equality sensing. Here $X = Y$ selects the third address, and $X \neq Y$, the fourth. In either type of branch operation, the quantity $-|X - Y|$ is available and can be read out subsequently if needed.

Arithmetic operations are checked by means of the arithmetic weighted count. This is a check number, associated with each operand and each result, and defined as the weighted sum of the digits of the number. The assigned weights are 1, 2, 4, 8, 16, 1, 2, 4, 8, 16, etc. Note that these weights differ from those assigned to the transfer weighted count described earlier. The algebraic sign of a number is not included in this weighted count since signs are checked by another method.

These counts are used to check the basic operations[1] $X + Y = Z$, $XY = Z$, and $X/Y = Z + R/Y$ by means of the identities:

$$(|Z_c - (X_c + Y_c)| + 31)_c \equiv 31 \qquad \text{(addition)}$$
$$(|(X_cY_c)_c - Z_c| + 31)_c \equiv 31 \qquad \text{(multiplication)}$$
$$(|(Y_cZ_c)_c + R_c - X_c| + 31)_c \equiv 31 \qquad \text{(division)}.$$

In the above identities, the subscript c indicates the arithmetic weighted count of the quantity to which it is affixed. In the last identity, R is the remainder.

To illustrate how the above relations are used by the machine, consider the addition check as an example. The sum $Z = X + Y$ is computed as described previously. Next X_c, Y_c, and Z_c are computed by means of an auxiliary arithmetic unit, the weighted count is taken, and the presence of 31 is determined by sensing.

Central Control.—The central control makes use of orders extracted from the memory to govern the machine. It contains facilities for making selections in the internal memory in accordance with addresses; for initiating either reading from the memory or writing into it, depending upon the position of the address in the order; and for initiating the proper arithmetic process as dictated by the operation code.

The operation of the machine is partly synchronous and partly asynchronous. The internal memory is completely synchronous, all pulses being timed from a central clock. The operations in the arithmetic unit are largely asynchronous. A complete machine cycle—viz., the time required to execute one complete order—must be a multiple of the memory word-cycle time of $45 \cdot \frac{1}{4} = 11.25$ microseconds. Memory selections and arithmetic operations are allocated only as much time as they actually require, to within one word-time; unnecessary memory selections are avoided. Thus, if the result of one arithmetic process is to be used solely as an operand in the next, it is not returned to the memory at all. Because of this so-called variable-cycle operation, certain arrangements of numbers and orders in the memory have distinct advantages for any given problem. Considerations of efficiency and logical planning dictate that, whenever it can be reasonably arranged, the result of an operation should be used as an operand in the succeeding operation; such forethought will often lead to a substantial reduction in computing time.

The nature of the central control can best be understood by reference to a list of the operations which it performs during a complete machine

cycle. These are:

(1) The first operand is selected under control of the first address.
(2) The line and word locations of the first operand in the internal memory are checked.
(3) The second operand is selected under control of the second address.
(4) Line and word locations of the second operand are checked.
(5) Transfer count check of the first order-word is made.
(6) Operational code is selected.
(7) Check-back from arithmetic unit is made to insure correct operation selection.
(8) End of arithmetic operation signals selection of memory position for result.
(9) Line and word locations of the position about to receive the result are checked.
(10) Selection of the new set of order-words is begun.
(11) The line and word locations of the new order-words are checked.
(12) Transfer count check of the second (present) order-word is made.

As explained earlier, all four of the addresses may not necessarily be present in an order; consequently, some of the above operations may be omitted.

The Problem Preparation Unit and the Printer: The problem preparation unit is used for the manual recording of problem input data on a magnetic tape by means of a keyboard. Either decimal numbers or binary numbers can be recorded on the tape. Ordinary decimal numbers set into the keyboard are recorded as binary-coded decimal numbers on the tape. Binary numbers are set into the keyboard in the octal (radix 8) notation. In this latter notation, each digital position corresponds to three binary digits. It can be seen that the conversions octal-binary and binary-octal are very elementary. The octal system is to be preferred to the binary since in the former notation only a third as many digits are required to represent a number.

All of the data which are recorded on the tape can be printed by means of an input page printer associated with the problem preparation unit. Thus a permanent, visually acceptable record is obtained of the information which is being introduced into the machine. The unit contains a device for generating transfer weighted counts since these must also be recorded on the magnetic tape.

Output numerical data recorded on magnetic tape by the machine are sent to either of two output printers. These data are normally in decimal form, though binary numbers could also be printed (actually in the octal notation rather than binary). Standard page printers, operating at speeds in excess of seven characters per second, can conveniently be used.

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[1] Subtraction, which is in reality nothing more than an addition after appropriate complementing, is checked by means of the addition identity; the complementing process is independently checked.

RECENT MATHEMATICAL TABLES

553[D].—WADE ELLIS & JOHN E. WALSH, **A.** *Table: $x \tan x$.* Electronics Research Laboratories, Air Materiel Command, Cambridge, Mass. (*ERL Report* no. E5016). Feb., 1948, iv, 19 p. 21.2 × 27.4 cm. **B.** *Table: $x \tanh x$.* Cambridge Field Station, Air Materiel Command, Cambridge, Mass. (*CFS Report* no. E5030). June, 1948, iv, 52 p. 21.2 × 27.4 cm.

In several problems considered in the Antenna Laboratory, Cambridge, Mass., the functions $f_1(x) = x \tan x$ and $f_2(x) = x \tanh x$ have arisen. Tables of these functions with the required range, interval of argument, and number of significant figures apparently do not exist; consequently, such tables were computed specifically to satisfy these needs. They were prepared in the form of reports with the belief that they might prove useful on a much wider basis.

A. Values of the function $f_1(x)$ are given for $x = [0(.001)1.570; 8D \text{ or } 8S]$. For the computation, values of $\tan x$ were taken from NBSCL, *Table of Circular and Hyperbolic Tangents and Cotangents for Radian Arguments*, New York, 1943. The range is immediately extended to $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$, by observing the identity $x \tan x = -x \tan(-x)$. The range may be extended without limit by use of the identity

$$x \tan x = [x/(x - n\pi)](x - n\pi) \tan(x - n\pi).$$

However, it is probably more convenient to compute values directly for arguments outside the range $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$.

It is believed that for many applications the arguments are sufficiently close together to render interpolation unnecessary. In cases where this is not true, reference should be made to the above mentioned table, p. xvii–xxii, or to any standard work on interpolation.

The parts of the table on pages 5 and 16 should be interchanged.

B. Values of the function $f_2(x) = x \tanh x$, which does not seem to have been previously tabulated, are given for $x = [0(.001)5; 8S]$. By virtue of the relation $(-x)\tanh(-x) = x \tanh x$ the range is immediately extended to $-5 \leq x \leq 5$. The computations were carried out at the Center of Analysis, Mass. Inst. of Technology.

For the computations, values of $\tanh x$, $0 \leq x \leq 2$ were taken from NBSCL, *Table of Circular and Hyperbolic Tangents and Cotangents for Radian Arguments*, New York, 1943. For $2 \leq x \leq 5$, HAYASHI, *Sieben- und mehrstellige Tafeln der Kreis- und Hyperbelfunktionen* . . ., Berlin, 1926, furnished the values of $\tanh x$, at interval .01, so that extensive subtabulation was necessary in order to provide values at .001 radian intervals of argument.

Extracts from text

EDITORIAL NOTES: There is a table of $x \tan x$ for $x = [0(.01)3.14, \frac{1}{2}\pi, \pi; 4\text{–}5S]$, in E. JAHNKE & F. EMDE, *Tables of Functions*, 1933 (p. 32–35); 1943, and 1945. Addenda, p. 32–35. In F. EMDE, *Tables of Elementary Functions*, Leipzig and Berlin, 1940, p. 125–126, there is a table of $x \tan x$ for $x = [0(10^g)1800^g; 4\text{–}5S]$. Up to $x = 900^g$ this table is copied in M. BOLL, *Tables Numériques Universelles*. Paris, 1947, p. 332; on p. 326–329, the JAHNKE & EMDE Table listed above is reprinted.

554[D].—[EMMA GIFFORD (1861–1936)], *Natural Tangents to Eight Decimal Places for Every Second of Arc from 0° to 90°.* First pirated American edition. Parker & Company, publishers, 241 E. Fourth St., Los Angeles, California, 1948, xii, 620 p. 17.2 × 26.7 cm. $15.00.

Of Mrs. Gifford's three published volumes of 8-place natural values of sines, tangents, secants, for every second of arc, the second, of tangents, appeared in instalments, beginning in 1920, and completed in 1927 (Manchester, viii, 620 p.). Of this edition the leaf (a), p. i–ii, is headed "Some discrepancies in Natural Tangents between Rheticus and Andoyer. The tangents from Rheticus are given in full and those figures in which Andoyer differs from him

are given underneath." (There are 41 10-place values in the list.) Leaves (b) and (c), p. iii–vi, are for half-title, and title pages. Leaf (d), p. vii–viii, contains a preface, and a list of 31 tangents which "have 0 in their 5th, 6th, 7th and 8th places." [As a matter of fact for two of the values thus listed, 36°7′46″ and 48°35′44″, PETERS does not agree with Mrs. Gifford in his similar 1939 volume for the trigonometric functions sin, tan, cot, cos.]

Mrs. Gifford's table for every second of arc is arranged quadrantally, horizontally up to 70°, each line corresponding to 10″, 10′ to a page, and then vertically 70°–90°, 6 minutes to a page. Differences, so called, given up to 84°, would have been much more advantageously replaced by proportional parts so that accuracy to hundredths of a second might have been readily found.

The preparation of the table was based on the *Opus Palatinum*, 1596, of Rheticus. This is a 10D canon, at interval 10″, with differences for all the tabular results, the six trigonometric functions arranged semiquadrantally.

From 45° to 60° (Mrs. G. remarks) some of the tangents, particularly those in the central columns of the page, read 1 too high in the 8th decimal place. The column headed 0 is correct throughout and so are the other tangents after 60°. Any discrepancies which may be noticed between cotangents in the first part and tangents in the second, are due to the fact that the second part has been corrected by ANDOYER's Table of Natural Tangents (Paris, 1916, at interval 10″), which was not published when this work was begun. The table fills exactly 620 pages.

In 1940 a second edition, offset print, of this table was prepared by the Scientific Computing Service, London (x, 620 p.), and 55 errors in the table of this second edition are listed on p. ix. Six other listed errors in the first edition (p. 157, 391(2), 617(2), 619) were corrected, but there are some 600 other known errors. This edition has been out of print since 1944.

Let us now turn to the edition under review. The 620 pages of the table differ only from the corresponding pages of the second edition of Gifford in the following respects: (1) the table has been photographically enlarged; (2) the 55 errata referred to above have been corrected in the table; (3) the page-numbers which, in the original, were at the bottom of each page in the center, are now (in a different fount of type) at the outer edges of each page; (4) the statement at the bottom of p. 615–620 ("When the whole number has 3 digits, only 7 decimal places are given") has been eliminated, and put in a preface, which is signed by R. B. HUEY. In this preface is given the list referred to above in (d), and the preface then concludes "Every effort has been made to eliminate errors in the preparation of this book but if any discrepancies are discovered we should appreciate being notified so that they may be corrected." From what we have indicated above, the "effort" made by Mr. Huey was confined to ordering the 55 corrections listed as errors in the second edition to be made in the new pirated edition, but hundreds of other errors persist.

The xii introductory pages of this Parker Co. edition are occupied as follows: p. i, title page. Under the title quoted above is the picture of an engineer using a theodolite. Under this picture is "R. B. Huey." Thus the title-page does not actually state that the table was by R. B. Huey. P. ii is blank; on p. iii we find "Copyright 1948 by Parker and Company, Los Angeles" and "Printed in the United States of America." The United States is the only country in the world which requires that a book from another country, to be copyrighted in the United States, must be printed there. Here, then, is a case of a publisher, who appropriated the work of an author, without permission or compensation, and published it as his own, without any reference whatever to the real author of the volume, or to her estate. A casual inspection of the volume would suggest that its author was Huey, but no definite statement to this effect is anywhere made.

On raising certain questions concerning the table Mr. Huey's reply to the reviewer contains the following statements: "The table was not computed by me as this work would have been superfluous. There were several tables of this sort computed in Europe in the past and this was taken from one that has been out of print, I understand, for more than twenty years, and was not available to the public. I would have preferred mentioning this

fact in the book but, because of certain technicalities, was advised not to. The only credit I wish to, or can, claim for the publication of the book is the fact that I spent a great deal of time and money to make it available to every Engineer and Surveyor in the United States at the lowest cost possible—and as soon as possible—as I realized the great need for such a book."

P. iv is blank; p. v, "About R. B. Huey" is signed "Parker and Company." On p. vi–vii are tables "Lengths of the arcs of circles to the radius 1" and "Minutes in decimals of a degree"—the only tables not in Gifford. P. viii is a reproduction of p. (a) referred to above. P. ix–x is an English preface, and p. xi–xii the same preface in Spanish. The tabular part of the strongly bound pirated volume is clearly reproduced on substantial paper.

In *MTAC* we have unfortunately had occasion more than once to refer to other appropriations in this country similar to that indicated above. In *MTAC*, v. 1, p. 112, we noted that a WITTSTEIN book was published in Chicago with the suggestion that a William W. Johnson was its author; on p. 8 and 112 of v. 1 we referred also to a JORDAN volume published in New York, as by a Wm. Chas. Müller.

Has the volume under review any real justification, so far as scientific needs of this country are concerned? Who would pay $15 for the volume described above, if he knows that at Ann Arbor, Michigan, he can for $20 purchase a copy of an outstanding, wholly accurate, work containing twice as many functions: J. T. PETERS, *Eight-Place Table of Trigonometric Functions for Every Sexagesimal Second of the Quadrant*, 1943? (See *MTAC*, v. 1, p. 147–148.) Here, in semi-quadrantal arrangements, sine, tangent, cotangent, cosine may be readily read off in separate columns.

R. C. A.

555[F].—A. GLODEN, "Sur les nombres terminaux des cubes dans le système de numération décimal," *Euclides*, v. 7, 1947, p. 393–397. 16.6 × 24.1 cm.

There are three tables giving the 1-, 2-, and 3-digit endings of cubes together with the linear forms of those numbers whose cubes have the given ending. The tables have 10, 63, and 505 lines respectively, and are arranged according to increasing values of the ending. This table, similar to others for squares, is useful for the rapid identification of non-cubes. A manuscript of this table was reported in *MTAC*, v. 2, p. 354, UMT 61.

D. H. L.

556[F].—H. GUPTA, "A table of values of $\tau(n)$," Nat. Inst. of Sci., India, *Proc.*, v. 13, 1947, p. 201–206. 17.5 × 24.9 cm.

The function $\tau(n)$, usually called Ramanujan's function, may be defined as the coefficient of x^{n-1} in the expansion of the 24th power of the product $(1 - x)(1 - x^2)(1 - x^3) \ldots$ and occurs in the arithmetic theory of elliptic functions. This table gives $\tau(n)$ for $n \leq 400$. There is no indication of the methods employed. Previous tables have been given by RAMANUJAN[1] ($n \leq 30$), LEHMER[2] ($n \leq 300$) and GUPTA[3] ($n \leq 130$). A remarkable table of $\tau(n)$ for $n \leq 1000$ by G. N. WATSON has been in press since 1942. Galley proof kindly supplied by its author has been read against the table under review. The agreement is perfect.

Extensive tables of $\tau(n)$ are useful in the study of the many properties and unsolved questions concerning this interesting function.

D. H. L.

[1] S. RAMANUJAN, "On certain arithmetical functions," Cambridge Phil. Soc., *Trans.*, v. 22, 1916, p. 174; *Collected Papers of Srinivasa Ramanujan*, Cambridge, 1927, p. 153.
[2] D. H. LEHMER, "Ramanujan's function $\tau(n)$," *Duke Math. Jn.*, v. 10, 1943, p. 483–492; see *MTAC*, v. 1, p. 183–184.
[3] H. GUPTA, "Congruence properties of $\tau(n)$," Benares Math. Soc., *Proc.*, s. 2, v. 5, 1943, p. 17–22; see *MTAC*, v. 2, p. 26–27.

557[F].—D. YARDEN & A. KATZ "Goremim chadashim shel mispare Fibonatsi" [New factors of Fibonacci numbers], *Riveon Lematematika*, v. 2, Jan. 1948, p. 35. 21.6 × 33.6 cm.

In this short note the factors 1069, 29717 and 27941 are announced for the 89th, 117th and 127th terms, respectively, of the Fibonacci series $u_n = 1, 1, 2, 3, 5, \cdots$. Besides these three factors and the ones already given in Yarden's previous table and supplement (*MTAC*, v. 2, p. 343; v. 3, p. 96) there are no further factors of the first 128 terms of the series u_n and $v_n = u_{2n}/u_n$ below 10^6. Of these 256 numbers 55 are as yet incompletely factored. As indicated above, the rank of apparition of 1069 is 89. In a previous table of the ranks of apparition of primes ≤ 1511 (*MTAC*, v. 2, p. 343) the rank of 1069 is given as 534. This error is noted in conclusion. D. H. L.

558[H, I, L, S, V].—AKAD. N., MATEMATICHESKIĬ INSTITUT imeni V. A. STEKLOVA, *Trudy*, v. 20, Leningrad & Moscow, 1947, 136 p. 15 × 24.6 cm. Paper covers, 10 roubles. Entirely in Russian.

This is a collection of works completed, during the years 1943–1944, in the Section on approximate calculations of the Mathematical Institute, edited by L. A. LIÛSTERNIK & V. A. DITKIN. The v. contains the following articles: I. ſa. AKUSHSKIĬ & V. A. DITKIN, "O chislennom reshenii uravneniĭa ſtsirkulíaſtsii koleblíûshchegosía kryla" [On a numerical solution of the equation of circulation of an oscillating wing], p. 7–38; an illustrative example worked out with details. I. ſa. AKUSHSKIĬ, "O nekotorykh voprosakh, svíazannykh s primeneniem schetno-analiticheskikh mashin" [On certain questions, connected with the application of analytical computing machines], p. 39–48. L. A. LIÛSTERNIK, "Zamechaniía k chislennomu resheniíû kraevykh zadach uravneniía Laplasa i vychisleniíû sobstvennykh znacheniĭ metodom setok" [Observations on the numerical solution of the boundary problems of Laplace's equation and on the computation of eigenvalues by the method of nets], p. 49–64. B. I. SEGAL, "Ob odnoĭ zadache teploprovodnosti" [On a certain problem of heat conduction], p. 65–76; an illustration worked out in detail. V. A. DITKIN, "Reshenie odnoĭ zadachi teploprovodnosti metodom operaſtsionnogo ischisleniía" [Solution of a certain problem of heat conduction by the method of operational computation], p. 77–86; this deals with the problem, by SEGAL, in the preceding art., and on p. 85–86 gives a table of values of $u(x)$ and $-v(x)$, defined by the relation $xH_1^{(1)}(x)/H_0^{(1)}(x) = u(x) + iv(x)$, where $H_0^{(1)}$ and $H_1^{(1)}$ are HANKEL's functions, for $x = [0(.02)1; 4D]$ and $[1(.1)9.9; 3D]$. L. ſa. NEĬSHULER, "O tabulirovanii funkſtsiĭ trech peremennykh" [On the tabulation of functions of three variables], p. 87–108; gives several schedules showing skeletons of tables. L. I. SHATROVSKIĬ, "Primenenie metoda Neĭshulera k sostavleniíû tabliſa vozdushnoĭ strel'by" [Application of Neĭshuler's method to the construction of tables for aerial gunnery], p. 109–112. L. ſa. NEĬSHULER, "Zametki po tabulirovaniíû" [Notes on tabulation], p. 113–116: 1. "Ob odnom variante tabliſsy umnozheniía" [On a certain variant of the multiplication table], p. 113–114; 2. "O tabulirovanii sistem lineĭnykh funkſtsiĭ mnogikh peremennykh" [On the tabulation of systems of linear functions of many variables], p. 115–116. V. M. PROSHKO, "Elektricheskiĭ pribor dlía resheniía sistemy sovmestnykh lineĭnykh algebraicheskikh uravneniĭ" [Electric apparatus for the solution of a system of simultaneous linear algebraic equations], p. 117–128. I. S. GRADSHTEIN, "Pribor dlía chercheniía grafika funkſtsii ot funkſtsii" [Apparatus for drawing a graph of a function of a function], p. 129–130. B. KHANOV, "Kinematicheskoe reshenie trekhchlennogo uravneniía" [A kinematic solution of a trinomial equation], p. 131–133. V. A. DITKIN, "Rabota seminara pri otdele priblizhennykh vychisleniĭ za 1943–1944 gg" [The work accomplished in the seminar at the section of approximate computations during the years 1943–1944], p. 134–135; this lists eleven authors, with corresponding titles, some of which are published in the present volume.

We note a few errata which occur in tables: p. 35, l. 12, *for* B^9, *read* B_{78}; p. 54, l. −7, −5, *for* 3.95, *read* 3.85; p. 55, l. 6, *for* 31.42, *read* 39.42; p. 55, l. −4, *for* 22.58, *read* 22.55; p. 86, l. 8, *for* 4.91, *read* .491. S. A. J.

559[K].—RAYMOND T. BIRGE, "Least squares' fitting of data by means of polynomials," *Rev. Modern Physics*, v. 19, 1947, p. 298–347. "Mathematical appendix," by J. W. WEINBERG, p. 348–360. 19.8 × 26.7 cm.

The purpose of the present paper is to set forth a critique of methods which have been developed for fitting a polynomial to empirical data by the method of least squares. The author characterizes the different schemes for polynomial approximation as follows: "The least-squares' solution of a polynomial may be given in (A) power-series form, or (B) factorial form. The solution may involve the explicit use of (1) power moments of the observations, (2) factorial moments of the observations, (3) the observations themselves, and (4) the various finite differences of the observations. Finally, the solution may be in (α) orthogonal polynomial form, (β) non-orthogonal polynomial form."

The author contrasts various solutions which have been given to this problem, beginning properly with the classical memoirs of P. L. CHEBYSHEV, which appeared just after the middle of the nineteenth century. The principal object of the paper is to give an exposition of the method of polynomial curve fitting, which was developed by BIRGE in collaboration with J. D. SHEA in 1927 and now slightly modified in the present paper. This method is compared with rival methods for solving the same problem. The paper is somewhat polemical as one may infer from the statement relative to the modified Birge & Shea procedure: "The method about to be presented represents what now appears to the writer the one most advantageous for the general use of physical scientists." There must be at least a dozen different techniques in existence, those, for example, of A. C. AITKEN, R. A. FISHER & F. YATES, S. M. KERAWALA, V. PARETO, E. JÄDERIN, C. JORDAN, F. E. ALLAN, and M. SASULY, to mention only a few of them. All have their merits and also certain defects, which depend in part upon the data which are to be fitted, and the form in which they are given. When the order of approximation is uncertain, then obviously it is better to employ a method which depends upon orthogonal polynomials, since the coefficients of added terms can be determined independently of those already computed. It would seem doubtful that any "best method" exists for all types of data and all problems.

The paper also devotes attention to the important problem of estimating the error of approximation. This is obtained from the sum of the squares of the residuals, a quantity which has been given previously in several places, and which is readily seen to be equal to the defect in Bessel's well-known inequality in the general theory of orthogonal functions.

The method of Birge & Shea, as well as the modified method given in the present paper, is designed to obtain the coefficients in the expression,

$$u_j(\epsilon) = \sum_{i=0}^{i} a_i T_i(\epsilon),$$

where $T_i(\epsilon)$ are polynomials, which satisfy the recurrence relationship

$$4(4i^2 - 1)T_{i+1}(\epsilon) - 4(4i^2 - 1)\epsilon T_i(\epsilon) + i^2(n^2 - i^2)T_{i-1}(\epsilon) = 0.$$

This difference equation goes back to Chebyshev and the polynomials to $i = 8$ were explicitly given by V. PARETO in 1899 (*Z. f. Schweizerische Statistik*, v. 1, 1899, p. 121–150). Two additional ones were later added by F. E. ALLAN in 1930 (R. Soc. Edinb., *Proc.*, v. 50, 1930, p. 310–320). The explicit formulae for the Chebyshev polynomials through $n = 10$ are reprinted in the present article.

The paper contains 14 tables of which five give illustrative data, one, namely Table III, gives the explicit forms of the polynomials, two give algebraic relationships between the various variables of the theory, and the remainder contain values of the various quantities used in the technique of fitting the polynomials to data. They are of small intrinsic interest, but necessary in the application of the theory. The last table gives the values of certain functions used in the theory of fitting the polynomials to data when finite differences, instead of direct observations, are used.

H. T. D.

560[K].—MARIAN C. JACKMAN, Tables 1, 2, 5, in C. I. BLISS, "Estimation of the mean and its error from incomplete Poisson distributions," Connecticut Agricultural Experiment Station, New Haven, *Bull.* no. 513, Jan. 1948, 12 p. 15.2 × 22.8 cm.

The phrase "random events" refers to points distributed either in time or in space. Typical examples for the latter are spores or bacteria in blood, animal litters in a field, weed-seeds among grass-seeds. The usual definition of complete randomness is that the probability p_k of having exactly k events in an interval, area, or volume of specified size is given by the POISSON distribution

$$p_k = e^{-m}m^k/k!,$$

where m is a positive constant, the so-called mean of the distribution. The problem then arises to estimate m from actual observations and to judge the reliability of the estimate. The usual methods are based on a complete enumeration of all cases where 0, 1, ⋯ events were observed. The author points out that very often it is difficult or impossible to obtain such a complete enumeration. Instead, the data are truncated at a class x so that only cases with 0, 1, ⋯, x events are enumerated and no information is available concerning cases with more than x events. The author describes a technique of estimating m and its error from such incomplete data. To facilitate the work he gives tables which are obtained from E. C. D. MOLINA, *Poisson's Exponential Binomial Limit*, New York, 1942.

Table 1 (p. 3) gives to 5D the cumulated values $P_x = p_0 + p_1 + \cdots + p_x$ for $x = 0,1,2,3$ and $m = .1(.1)6$. Table 2 (p. 4) gives $p_x/(1 - P_x)$ within the same range. This table is to 6D or 6S. Table 5 (p. 8) gives values obtained from these tables and useful for the estimation of the error: this table is of interest only to specialized statisticians. Other tables refer to empirical observations.

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561[K].—T. L. KELLEY, *The Kelley Statistical Tables. Revised 1948.* Cambridge, Mass., Harvard Univ. Press. 1948, x, 223 p. 21.4 × 27.8 cm. $6.00. The first edition was published in 1938; see *MTAC*, v. 1, p. 151–152.

It is not the purpose of this book to provide an extensive selection of many tables used by statisticians. The main part of the book is devoted to tables of the normal distribution arranged in a handy form which will in many cases increase the usefulness and convenience of the tables. All other tables are more or less auxiliary tables designed so that maximum information can be extracted from the main table. A special feature of the book is the attention paid to interpolation errors. Careful instructions for direct and inverse interpolation are given. With the natural exception of the tables of interpolation coefficients all columns in this book indicate at the bottom the corresponding maximum two- and three-point interpolation errors for both direct and inverse interpolation. This feature will be appreciated by many users.

T.I, the main table, extends over p. 38–137 with a supplement on p. 36 and extensive explanations in the first part of the book. Let $z = (2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}$, and $p = \int_{-\infty}^{x} z\,dx$, $q = 1 - p$. The columns of the table are p, x, z, $(pq)^{\frac{1}{2}}$, $(1 - p^2)^{\frac{1}{2}}$, $(1 - q^2)^{\frac{1}{2}}$, q in the order named. The argument $p = .5(.0001).9999$ and $q = .5(-.0001).0001$. Between them the two arguments cover the entire range: if q is used in the meaning of p, then x has to be taken with the negative sign. All tabulated values are to 8D. Linear interpolation is sufficient over almost the entire range of the table. Values at the extreme end of the table are of importance to statisticians and are provided in a supplementary table on p. 36. It gives p, x, z, q for the ten values p = .9999 5, .9999 9, .9999 95, .9999 99, .9999 995, .9999 999, .9999 9995, .9999 9999, .9999 9999 5, .9999 9999 9. The number of significant places for x and z decreases but is never less than 5.

For the computation of the table various methods and checks were used. Values were computed to 14D, and any inaccuracy of the last digit in the table should be due to a rounding error in the 11th place or better. Tables of the familiar Lagrange interpolation coefficients in the usual arrangement are provided as follows. Three-point coefficients in **T. II** (p. 138–187) for $p = [0(.0001).5; 5D]$. Four-point coefficients in **T. III** (p. 188–197) for $p = [0(.001).5; 7D]$. Six-point coefficients in **T. IV** (p. 198–199) for $p = [0(.01).5; 10D]$. Finally, eight-point coefficients in **T. V** (p. 200) for $p = [.1(.1).9; 11D]$.

T. VI is devoted to the chi-square distribution familiar to all statisticians. For pure mathematicians the contents may be described in terms of the incomplete gamma function rather than the statistical form. Define $F_n(x) = 1/\Gamma(\frac{1}{2}n) \int_{x^2}^{\infty} t^{\frac{1}{2}(n-2)} e^{-t} dt$. (This notation is not generally used.) On p. 202–203 we find double-entry tables. The independent variable is called $\chi/n^{\frac{1}{2}}$ and ranges from 0 to 4.1 in steps of .1. The second column is χ^2/n, and contains exact values (2D). In the successive 15 columns $F_n(\chi^2)$ is tabulated to 4D for $n = 1(1)10$, 12, 15, 19, 24, 30. For larger values of n the function can be approximated by the normal distribution of **T. I** and detailed rules are provided how this should be done. Two-way interpolation in **T. VI** is described and maximum errors tabulated at the bottom of the table and on p. 201.

T. VII (p. 204–221) gives $N^{\frac{1}{2}}$, $(10N)^{\frac{1}{2}}$, $N^{\frac{1}{3}}$, $(10N)^{\frac{1}{3}}$, $(100N)^{\frac{1}{3}}$, $\ln N$ for $N = [1(.01)10; 8S]$.

T. VIII (p. 222) gives $\theta_i = 3(2)^{-\frac{1}{3}} - 2^{\frac{1}{3}}/(3i)$ for $i = [1(1)100(10)200(20)300(50)500(100)1000, 10000; 7D]$. This function is used to reduce certain distributions of interest to statisticians to the normal distribution.

T. IX (p. 223) collects a few frequently needed constants and their logarithms, to 15D.

WILL FELLER

562[K].—K. L. NIELSEN & L. GOLDSTEIN, "An algorithm for least squares," *Jn. Math. Phys.*, v. 26, 1947, p. 120–132. 17.4 × 25.4 cm.

In this paper the authors make use of an algorithm designed by P. D. CROUT (AIEE, *Trans.*, v. 60, 1941, p. 1235–1241) for the solution of linear systems of equations. The algorithm is adapted to the problem of fitting a polynomial of the form,

$$y = \sum_{i=0}^{n} a_i x^i,$$

to a set of values of y, $\{y_j\}$, given at the equally spaced abscissa values: $x_j = x_1 + (j-1)h$, $j = 1(1)m$.

Normalizing the data by introducing the variables: $X_j = x_j - \bar{x}$, $Y_j = y_j - \bar{y}$, where we abbreviate,

$$\bar{x} = \frac{1}{m}\sum_{j=1}^{m} x_j, \qquad \bar{y} = \frac{1}{m}\sum_{j=1}^{m} y_j,$$

the authors reduce the problem to finding the values of b_i in the equation,

$$Y = \sum_{i=0}^{n} b_i X^i,$$

which will give the best fit to the observed values: X_j and Y_j, in the sense of least squares.

The sum of the squares of the residuals:

$$R_j = \sum_{i=0}^{n} b_i X_j^i - Y_j,$$

is then minimized by the method of least squares, which leads to the customary normal equations which must be solved for the coefficients, b_i. It is to this problem that the algorithm of Crout is then applied.

Introducing a new variable α_i defined by the equation: $b_i = \alpha_i k^{-i}$, $i = 0(1)n$, and a function, S_z, defined as follows:

$$S_z = \sum_{k=1}^{r-1} k^z, \quad \text{if } m = 2r-1; \qquad = \tfrac{1}{2}\sum_{k=1}^{r} (2k-1)^z, \quad \text{if } m = 2r;$$

the authors then compute the following observable quantities:

$$c_1 = 0, \qquad c_i = \tfrac{1}{2}S_2^{-1}h^{-i+1}\sum_{j=1}^{m} X_j^{i-1}Y_j.$$

The solution of the problem is then finally given in terms of the following iterative expressions:

$$\alpha_i = K_{i+1} - \sum_{j=i+2}^{n} \alpha_j A_{i+1,j+1}, \qquad i = 0(1)n,$$

$$K_i = \left[c_i - \sum_{j=1}^{i-1} K_j A_{ij} \right] / A_{ii}, \qquad i = 1(1)n+1,$$

where A_{ij} is the general element of the solving matrix for the linear system which defines the α_i. It is observed that $A_{ij} = 0$ if $i + j$ is an odd number.

To facilitate the computation a table is given for the values of A_{ij}, $i = 1(1)5$, $j = 3(1)5$, and for values of $m = 5(1)100$, computed to 6D. A table is also included over the same range of m for values of the function S_z, which are given exactly.

H. T. D.

563[L].—W. R. BENNETT, "Distribution of the sum of randomly phased components," *Quart. Appl. Math.*, v. 5, Jan. 1948, p. 385–393; table and graph, p. 387–388. 17.7 × 25.4 cm.

Table I gives the values, 4–6D, of $\phi_n(k) = n\int_0^\infty J_0(kz)J_1(z)J_0^{n-1}(z)dz$, for $n = 6(1)10$, $k = 1(1)10$ and a few scattered values, for $n = 6$, $k = 5.8$; $n = 7$, $k = 6.6$; $n = 8$, $k = 7.2$, 7.5; $n = 100$, $k = 4(3)10(5)30$. $\phi_n(k)$ is equal to the probability that the envelope of n sine waves of equal amplitude exceeds k times the amplitude of one.

Extracts from text

564[L].—C. J. BOUWKAMP, "Concerning a new transcendent, its tabulation and application in antenna theory," *Quart. Appl. Math.*, v. 5, 1948, p. 394–402.

$$E(z) = C(z) + iS(z) = \int_0^z (1 - e^{-it})dt/t,$$

where

$$S(z) = Si(z) = \int_0^z \sin t\,dt/t, \qquad C(z) = \gamma + \ln z - Ci(z) = \int_0^z (1 - \cos t)dt/t.$$

$$E_1(z) = \int_0^z E(t)dt/t = \int_0^1 \int_0^1 (1 - e^{-iszt})dsdt/st.$$

The function $E_1(z)$ was encountered in antenna theory but may possibly be of some value in other fields as well. The functions $\alpha_1(x)$ and $\alpha_2(x)$ occurring in Hallén's antenna theory were found to be expressible in terms of the functions $E(x)$ and $E_1(x)$ as follows:

$$\alpha_1(x) = \tfrac{1}{2}e^{4ix}E(4x) - \cos x\, E(2x)$$

$$\alpha_2(x) = -\alpha_1(x)[\ln 4 + E(2x)] - \tfrac{1}{4}\cos x E^2(2x) + 2i \sin x E_1(4x) + \cos x\,[E_1(4x) - 2E_1(2x)].$$

On p. 401 are tables of the real and imaginary parts of $E_1(z)$ for $z = [0(.2)20; 6D]$, and on p. 402, tables of real and imaginary parts of α_1 and α_2 for $x = [0(.1)5; 4D]$. The tables of

$\alpha_1(x)$ show only small differences from those of R. KING & F. G. BLAKE in Inst. Radio Engin., *Proc.*, v. 30, 1942, p. 337. The tables for α_2 are in good agreement with the corresponding two-decimal values obtained by graphical integration in C. J. BOUWKAMP, *Physica*, v. 9, 1942, p. 618–619.

Extracts from text

565[L].—ENZO CAMBI, "Complete elliptic integrals of complex Legendrian modulus," *Jn. Math. Phys.*, v. 26, p. 234–245, 1948. 17.4 × 25.3 cm.

These interesting tables have arisen from a problem in differential equations and not from any problem of quadrature or of elliptic functions as such. For the purpose of this review, it appears simpler to reverse the author's order, by first describing his six tables from an elliptic standpoint, and then briefly outlining their relation to a differential equation.

All six tables may be described as tables of functions of a squared elliptic modulus, $z = k^2$, when $|z| = 1$; say $k^2 = -e^{ix}$, where x is real, and it suffices to consider $0 \leq x \leq \pi$. The functions involved are the usual complete integrals K, K', E, E' or combinations thereof, the definitions being

$$K(z) = \int_0^{\pi/2} (1 - z \sin^2 \phi)^{-\frac{1}{2}} d\phi, \qquad K'(z) = K(1 - z),$$

$$E(z) = \int_0^{\pi/2} (1 - z \sin^2 \phi)^{\frac{1}{2}} d\phi, \qquad E'(z) = E(1 - z).$$

These four integrals may all be computed either from the scale of arithmetic-geometric means of $(1, k')$, or from the scale of a.g. means of $(1, k)$, or from any expedient combination of the two processes. The fact that k is complex is no hindrance, apart from the usual slight arithmetical complication involved in working with complex numbers as compared with real ones. There is no difficulty in taking suitable determinations of sign in computing the square roots giving geometric means.

In the following descriptions of **T. I–VI**, it is to be understood that $k^2 = -e^{ix}$.

T. I (p. 238) gives $|K'|$ to 10D with δ^2 for $x = 0(0^q.1)2^q$, where $1^q = 1$ quadrant $= \frac{1}{2}\pi$. It is easily seen that the phase of K' is $\frac{1}{4}(\pi - x)$, which is also tabulated.

T. II (p. 240) gives $|K|$ to 10D with δ^2 and phase K to 7D of 1^q without differences for $x = 0(0^q.1)2^q$.

Since K is infinite at $x = \pi$ ($k^2 = 1$), **T. III** (p. 241) gives $|K|$ and $|K'|$ to 10D, phase K to 7D of 1^q, and phase K' exactly, all without differences, for $x = 1^q.9(0^q.01)2^q$.

T. IV (p. 242) gives $|k'|$, phase k', $R(K/K')$, $|K/K'|$ and $-\log |q'|$, all to 10D (except that phase k' is exact) and without differences, for $x = 0(0^q.1)1^q.9(0^q.01)2^q$. $R(K/K')$ denotes the real part of K/K'. $I(K/K')$, the imaginary part of K/K', does not need tabulation, since it is $\pm\frac{1}{2}$; with K and K' as tabulated in **T. I** and **II**, it is $-\frac{1}{2}$. It follows that $q' = e^{-\pi K/K'}$ is purely imaginary.

T. V (p. 243) gives $v = \frac{1}{2} \tan \frac{1}{2}x$ to 7D, $|k'K|$ to 10D, phase $(k'K)$ to 7D of 1^q, and $R(k'K)$, $I(k'K)$ to 7D, all without differences, for $x = 0(0^q.1)1^q.9(0^q.01)2^q$. The last two columns of this table give solutions of a differential equation, as will be explained below.

T. VI (p. 244) gives to 10D, without differences, the real and imaginary parts of $e^{\frac{1}{2}i(\pi - x)}(1 - E'/K')$ for $x = 0(0^q.1)2^q$. The function was convenient to calculate, and E' may evidently be derived from **T. I** and **VI** by elementary computation.

E may then be found from Legendre's relation

$$EK' + E'K - KK' = \tfrac{1}{2}\pi.$$

Cambi gives no table involving E.

The reason for the varying number of decimals given for the various functions is that the accuracy of those functions whose computation requires trigonometrical tables was limited by the use of 7-figure tables.

It is evident that the tables, with the help of some elementary computation if E and E' are required, define with considerable numerical precision the chief functions of the modulus

when k^2 lies on the unit circle centered at the origin. (For other complex values of k^2, see the diagrams on p. 46, 74, 76 of the 1945 Dover edition of JAHNKE & EMDE.)

It remains to give some idea of the utility of **T. V** in connection with a differential equation. Cambi's paper should be consulted for further details. $K(z)$ and $K'(z)$ are well known to be independent solutions of the hypergeometric equation

$$z(1-z)d^2y/dz^2 + (1-2z)dy/dz - \tfrac{1}{4}y = 0.$$

Putting $z = t/(t-1)$, and transforming the differential equation, we find that $K(t/t-1)$ and $K'(t/t-1)$ are independent solutions of

$$t(1-t)^2d^2y/dt^2 + (1-t)^2dy/dt + \tfrac{1}{4}y = 0.$$

Putting further $t = -e^{ix}$, and again transforming, we find that independent solutions of the equation

$$8(1+\cos x)d^2y/dx^2 + y = 0 \tag{1}$$

are $K(\frac{1}{2} + \frac{1}{2}i\tan\frac{1}{2}x)$ and $K'(\frac{1}{2} + \frac{1}{2}i\tan\frac{1}{2}x) = K(\frac{1}{2} - \frac{1}{2}i\tan\frac{1}{2}x)$. But these are complex conjugates, hence the real and imaginary parts of $K(\frac{1}{2} + \frac{1}{2}i\tan\frac{1}{2}x)$ are independent solutions of (1). These are the quantities tabulated in the last two columns of **T. V**, since it is easily shown that

$$K(t/t-1) = \sqrt{1-t}\,K(t) = k'K \text{ of Table V.}$$

Since (1), like Hill's equation, belongs to the class of linear differential equations with periodic coefficients, the result is of obvious interest.

A. FLETCHER

566[L].—H. B. DWIGHT, "Table of roots for natural frequencies in coaxial type cavities," *Jn. Math. Phys.*, v. 27, p. 84–89, 1948, 17.4 × 25.3 cm.

On p. 85–87 are tables relating to the roots, $x_n^{(s)}$, of

$$J_n(x)N_n(kx) - J_n(kx)N_n(x) = 0. \tag{1}$$

$(k-1)x_n^{(s)}$ is given to 2–6S for $n = 0(1)3$, $s = 1(1)5$, and for values of k always extending from $k = 1$ to some point in the sequence

$$1(.1)1.6(.2)2(.5)4(1)6(2)20(5)50.$$

For $n = 0,1,2,3$, the value of k does not for any s exceed 50,50,2,4 respectively. $(k-1)x_n^{(s)}$ is tabulated instead of $x_n^{(s)}$ for convenience in interpolation.

On p. 88–89 are tables relating to the roots, $x_n'^{(s)}$, of

$$J_n'(x)N_n'(kx) - J_n'(kx)N_n'(x) = 0; \tag{2}$$

a heading to this effect would have been useful. $x_n'^{(1)}$ and $(k-1)x_n'^{(s)}$, $s = 2(1)6$, are given to 2–6S for $n = 1(1)3$ and for values of k always extending from $k = 1$ to some point in the sequence given above. For $n = 1,2,3$, the value of k never exceeds 50,4,4 respectively. Values of $x_0'^{(s)}$ are not tabulated, since $x_0'^{(s)} = x_1^{(s)}$. The tables include roots of (2) not derivable at all from the MCMAHON asymptotic formulae.

Some tabulated values of (1) coincide with values in the well-known tables of KALÄHNE (reproduced in all editions of JAHNKE & EMDE) and others; see FMR, *Index*, p. 268, and *MTAC*, v. 2, p. 37.

As Dwight points out, numerical values of the roots of the prime equation (2) do not seem to have been published previously, though graphs for $s = 1$ were given in R. TRUELL, "Concerning the roots of $J_n'(x)N_n'(kx) - J_n'(kx)N_n'(x) = 0$", *Jn. Appl. Phys.*, v. 14, 1943, p. 350–352.

It so happens that the same number of *Jn. Math. Phys.* has (v. 27, p. 37–48) a paper by M. KLINE, "Some Bessel equations and their application to guide and cavity theory," which contains graphs of roots of

$$J_n(x)N_n(\rho x) - J_n(\rho x)N_n(x) = 0$$

and

$$J_n'(x)N_n'(\rho x) - J_n'(\rho x)N_n'(x) = 0.$$

Kline takes $0 \leq \rho \leq 1$, but the reader will have no difficulty in following the correspondence between Dwight's tables and Kline's graphs; the latter cover some ranges not included in Dwight's tables.

A. FLETCHER

567[L].—MARION C. GRAY & S. A. SCHELKUNOFF, "The approximate solution of linear differential equations," *Bell System Tech. Jn.*, v. 27, Apr. 1948, p. 350–364. 15 × 22.9 cm.

For the differential equation $y'' + x^{-1}y' + y = 0$, $0 < x \leq \infty$, p. 362, $y(x) = J_0(x) - IY_0(x)$, and at infinity $\bar{W}_0 = (1 + i)e^{-ix}(\pi x)^{-\frac{1}{2}}$, $\bar{W}_1 = \frac{1}{2}(1 + i)e^{ix}(\pi x)^{-\frac{1}{2}}[Ci(2x) - i(Si(2x) - \frac{1}{2}\pi)]$. There is a table for $x = .6(.2)1(1)10$, or $y(x)$, $\bar{W}_0$, $\bar{W}_0 + \bar{W}_1$, and for $x = .6(.2)1$, of $\bar{W}_0 + \bar{W}_1 + \bar{W}_2$. Graphs of $\bar{W}_0$ and $\bar{W}_0 + \bar{W}_1$, p. 360.

For the differential equation $y'' + x^{-1}y' = (x^{-2} - 1)y$, $1 \leq x \leq 3$, $y(1) = 1$, $y'(1) = 0$, p. 363, a solution using Picard's method and the integraph was described by T. C. FRY, Intern. Congr. Mathems., Toronto, *Proc.*, v. 2, 1928, p. 405–428. For the accurate solution $y = 1.4034J_1(x) - 0.3251Y_1(x)$, $x^{\frac{1}{2}}W_0 = \cos\beta(x - 1) + (2\beta)^{-1}\sin\beta(x - 1)$, and $x^{\frac{1}{2}}W_1 = (4\beta)^{-1}\int_1^x(3u^{-2} - 1)[\cos\beta(u - 1) + (2\beta)^{-1}\sin\beta(u - 1)]$, $\sin\beta(x - u)du$, $\beta = \frac{1}{2}\sqrt{3}$. There are tables of y, W_0, and $W_0 + W_1$, and of the third and eighth Picard approximations y_3 and y_8 taken from Fry's paper, p. 410, for $x = [1(.2)3; 3D]$.

Extracts from text

568[L].—E. L. KAPLAN, "Auxiliary table for the incomplete elliptic integrals," *Jn. Math. Phys.*, v. 27, p. 11–36, 1948. 17.5 × 25.4 cm.

It is well known that interpolation in LEGENDRE's classic tables of the incomplete elliptic integrals $F(\theta, \phi)$ and $E(\theta, \phi)$ is difficult or impossible when both the modular angle $\theta = \sin^{-1} k$ and the amplitude ϕ are near 90°. Kaplan's important paper gives (i) tables of auxiliary functions for the determination of $F(k, \phi)$ and $E(k, \phi)$ in the case mentioned, (ii) lists of some of the errata in Legendre, based partly on the results of previous workers and partly on a certain amount of original checking and of recomputation using (i).

Kaplan puts $r = k'/k = \cot\theta$, $x = \cos\phi$, and uses auxiliary functions f and e which may be defined by

$$K - F(k, \phi) = 2\pi^{-1}K' \sinh^{-1}(x/r) + x(r^2 + x^2)^{\frac{1}{2}}f,$$
$$E - E(k, \phi) = 2\pi^{-1}(K' - E')\sinh^{-1}(x/r) + x(r^2 + x^2)^{\frac{1}{2}}e.$$

His paper should be consulted for the series giving $f(r, x)$ and $e(r, x)$ and for further algebraical developments. On p. 20–35 he tabulates f and e to 10D for $x^2 = \cos^2\phi = -.005(.005) + .160$ and $r^2 = k'^2/k^2 = -.005(.005) + .160$, the negative arguments $-.005$ being included to facilitate interpolation. He remarks that fourth differences may be neglected, and that linear and quadratic interpolation is valid to about 6D and 8D respectively. On p. 36 Kaplan tabulates $2K'/\pi$ and $2(K' - E')/\pi$ to 12D, with modified second differences, for $r^2 = -.005(.005) + .160$. Kaplan's tables appear to the reviewer to be of very great value.

With respect to errors in Legendre, Kaplan gives two main lists. The first, on p. 17, relates to errors in PEARSON's Table I, which photographically reproduces Legendre's Table I of 1826, and gives log K and log E to 12–14D for $\theta = 0(0°.1)90°$. Kaplan gives 2 errors in log K, 6 in log E, 14 in the differences, of various orders, of log K, and 11 in the differences of log E. The 8 errors in the function values include 4 not previously published (but known to the reviewer). Two cases of slightly defective type in Pearson's reprint are noted. The errors in differences will form useful material, but such as relate to final digits are of little importance, since the reviewer shows elsewhere in this number that the function

values require many more corrections than Kaplan gives. Until the values of $\log K$ and $\log E$ have been satisfactorily corrected, one is not in a position to consider any but quite gross errors in the differences. In each case Kaplan indicates whether or not the errors in his list occur in Legendre's Table I of 1816.

Kaplan's second list of Legendre errata, on p. 18–19, is far more important. It relates to Pearson's Table II which photographically reprints Legendre's Table IX of 1826, and gives $F(\theta, \phi)$ and $E(\theta, \phi)$ to 9–10D for $\theta = 0(1°)90°$, $\phi = 0(1°)90°$. Some supplementary information is given on p. 15. Kaplan appears to the reviewer to list all previously known errors in Pearson's Table II, and he gives many more found by himself, partly through recomputation of some values by his own tables. Kaplan indicates in all cases whether or not the error is also present in the 1816 edition of Legendre's Table IX (photographically reprinted by Potin and by Emde). It need only be noted that there are three known errors in Legendre's 1816 values of $E(\theta, \phi)$ which were corrected in 1826, and therefore do not appear in Kaplan's list. These are:

θ	ϕ	*For*	*Read*
51°	23°	0.33502	0.39502
82	14	0.24195	0.24196
86	2	0.03489 9351	0.03489 9531.

On account of the great importance of Kaplan's list of errors in $F(\theta, \phi)$ and $E(\theta, \phi)$, it is reproduced by kind permission in the Guide to Tables of Elliptic Functions elsewhere in this number.

A complete list of the errors in Legendre's double-entry tables has never been given, and remains a desideratum, but the list under review goes further than any previous list.[1]

A. Fletcher

[1] Errors in Legendre have previously been pointed out in:
K. Bohlin, *Tables des Fonctions Elliptiques*, Stockholm, 1900, p. 3.
C. J. Merfield, "Traité des Fonctions Elliptiques (Legendre). Errors in Tome II," *Astron. Jn.*, v. 30, p. 190, 1917.
S. P. Glazenap, *Matematicheskie i Astronomicheskie Tablitsy* [Mathematical and Astronomical Tables], Leningrad, 1932, p. 214.
N. S. Samoĭlova-Îakhontova, *Tablitsy Ellipticheskikh Integralov* [Tables of Elliptic Integrals], Moscow and Leningrad, 1935, p. 6.
C. Heuman, "Tables of complete elliptic integrals," *Jn. Math. Phys.*, v. 20, 1941, p. 127–206, 336; also sheet of remarks, Stockholm, 1941.
G. Witt, "Sechs Berichtigungen . . . ," *Z. angew. Math. Mech.*, v. 21, 1941, p. 254; reported by F. Emde.
R. C. Archibald, *MTAC*, v. 2, p. 136–137, 181, 1946.

569[L].—V. Kourganoff, "Sur les fonctions $K_n(x) = \int_1^\infty e^{-xt} dt/t^n$ et certaines intégrales qui s'y rattachent," *Annales d'Astrophysique*, v. 10, 1947, p. 282–299. 21.8 × 27.3 cm.

$$I_{pqn}(a) = s!F(n, 1, n + s + 1, -p/a)/[(n + s)(p + a)^s a]$$

On p. 295–296 are given the values of $I_{1,q,n}(1)$ and $I_{2,q,n}(1)$ for $n = 1(1)8$; also of $I_{1,1,n}(1)$, $I_{2,1,n}(1)$, $I_{1,2,n}(1)$, $I_{1,3,n}(1)$, $I_{1,4,n}(1)$, for $n = 1(1)7$.

On p. 298–299 are values of $K_2(x)$, $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, $x = [0(.02)2;4D]$, except $\phi_2(x)$, 4S, $1 < x \leq 2$; $\phi_0(x) = 2K_2(x)$, $\phi_1(x) = \frac{1}{2} - 2K_1(x)$, $\phi_2(x) = \frac{1}{4}x + 4K_2(x)$.

570[L].—S. O. Rice, "Reflections from circular bends in rectangular wave guides—matrix theory," *Bell System Techn. Jn.*, v. 27, Apr. 1948, p. 305–349, tables p. 343–344. 15 × 22.9 cm.

$$I_s = \pi^{-1}\omega^s[u^{-1} - (-1)^s v^{-1} - \pi^{-1} s J_s],$$
$$J_s = \pi[Si(sv) - Si(su)]\cos su - \pi[Ci(sv) - Ci(su)]\sin su,$$
$$K_s = (\rho_1/a)\{[Ci(sv) - Ci(su)]\cos su + [Si(sv) - Si(su)]\sin su\},$$
$$I_0 = [(2\rho_1/a)^2 - 1]^{-1},\ K_0 = (\rho_1/a)\log(1 + \pi/u),$$
$$u = \rho_1\pi/a - \tfrac{1}{2}\pi,\ v = \rho_1\pi/a + \tfrac{1}{2}\pi.$$

There are tables of I_s, J_s, K_s for $s = 0(1)6$, $\rho_1/a = [.5(.1)1.5(.5)2.5; 5D]$, except for $\rho_1/a = 2,2.5$, the values of I_5, I_6, J_5, J_6, K_5, K_6 are omitted; also values of J_s, $s = 1(1)6$, for $\rho_1/a = .5$.

Extracts from text.

571[L, M].—H. TIETZE, "Zusammanstellung einiger Werte des Integrallogarithmus," Akad. d. Wiss., Munich, *math.-naturw. Kl., Sitzb.*, 1947, p. 47–50. 14.5 × 22.6 cm.

Two small tables are given of the logarithmic integral $Li(x)$. One is a table to 8S for

$$x = [2(1)10(10)100(100)1000(1000)10000(10000)100000]$$

and is due to HANS CARL HAMMER. This table, with 25 of its 90 values quoted, is described as UMT **54** in *MTAC*, v. 2, p. 280, where also will be found references to previous tables of this function. The other table is 5–6S for the above values of x, and also for $x = 200000$ and 300000, and is due to LEONHARD WEIGAND. The last half of the table was produced by actual quadrature of the integral. The agreement of the two tables is very good. The reviewer has noted two errata:

Hammer, $x = 7$ *for* 4.7570508, *read* 4.7570518
Weigand, $x = 3 \cdot 10^8$ *for* 26086.4, *read* 26086.7.

More accurately $Li(300000) = 26086.6920$.

The first of these errors shows that the corresponding entries in DEMORGAN's table, 1842, p. 663, and SOLDNER's table, 1809, p. 48, are also in error.

D. H. L.

572[M].—S. O. RICE, "Statistical properties of a sine wave plus random noise," *Bell System Techn. Jn.*, v. 27, Jan. 1948, p. 109–157. 15 × 22.9 cm.

On p. 152–153 is a table of $Ie(k, x) = \int_0^x e^{-u} I_0(ku)du$ for $x = [0(.2)5(.4)9(1)15, \infty; 4D]$, $k = 0(.2)1$; and for $x = [15(1)20, \infty; 4D]$, $k = .86, .96, 1$. Also for $k = .9$, values for $x = [10(1)20, \infty; 4D]$.

573[M].—D. E. THOMAS, "Tables associated with a semi-infinite unit slope of attenuation," *Bell System Tech. Jn.*, v. 26, 1947, p. 870–899.

In this paper are tables of $B(x) = \pi^{-1}\int_0^\infty \ln\left|\frac{1+t}{1-t}\right| dt/t = 2\pi^{-1}\int_0^\infty \tanh^{-1} t\,dt/t$, and of $(180/\pi)B(x)$, for the following values of x and $1/x$: [0(.001).996(.0003).998(.0001)1; 5S]. These tables appreciably extend those of CORRINGTON, *MTAC*, v. 2, p. 218.

574[S].—HAROLD J. PEAKE, *An Aid to the Calculation of Electric Field Strengths in the Interference Region.* N. R. L. Report no. R-3222. Washington, D. C., Naval Research Laboratory, March 1948, vi, 140 p. 20.3 × 26.7 cm.

If E = electric field strength, E_0 = free space field strength at unit distance, d = distance from transmitting antenna, D = divergence factor, $|R'|$ = magnitude of reflection coefficient, $\alpha = \theta - C$ (where θ = phase difference between direct and reflected waves due to difference in path length, C = phase shift upon reflection), then

$$E = (E_0/d)I^{\frac{1}{2}},\ \text{where}\ I = 1 + (D|R'|)^2 - 2D|R'|\cos\alpha.$$

If db_t = attenuation of electric wave in decibels, λ_m = wavelength in meters, and d_{nm} = distance in nautical miles, then

$$db_t = 10 \log I + 20 \log \lambda_m - 20 \log d_{nm} - 83.04.$$

The tables, p. 5–140, are of I and $\log I$ to at least 3S(4–5D) for $D|R'| = .001(.001)1$, for $\cos \alpha = -1, 0, +1$; and $D|R'| = .01(.01)1$, for $\cos \alpha = .01(.01).99$.

Extracts from introductory text.

575[U].—France, Service Technique Aéronautique (prepared by M. L. Chamaleix), *Table de Hauteur et d'Azimut, A.A.F. Type 10;* an official publication of the Armée de l'Air Française (A.A.F.), not available for general sale. Eleven volumes have so far been issued and three more are in the press. "Édition 1946," distinguished by silver stars on the outside cover, consists of seven volumes, 16.5 × 23.5 cm., bound; Vol. 0, 223 p.; 1, 239 p.; 2, 3, 4 and 5, each 231 p.; 6,223 p. The other volumes are the same size and contain: Vol. 2, 303 p.; 3, 319 p.; 4, 323 p.; 5, 311 p.

The French air almanac, *Ephémérides Aéronautiques*, was redesigned as from January 1948 to become almost identical with the *American Air Almanac*. The list of the 72 brightest stars on the cover was divided into groups similar to those in the British and American air almanacs; but this grouping is a consequence of the use of the *Astronomical Navigation Tables* (reprinted in U.S.A. as H.O. 218; see RMT **106** and **448**), in which separate tabulations are given for the 22 brightest stars, and would be inappropriate unless similar tables were in use. Enquiries through the office of the *Connaissance des Temps* resulted in the verification of the existence of these volumes. I am indebted to le Directeur du Service Technique Aéronautique (STA) of the French Ministère des Forces Armées not only for a complete set of the tables so far issued but also for permission to include here the short account of these tables, prepared by M. J. Jacq of the STA.

In general appearance the tables are very similar to the *Astronomical Navigation Tables* (ANT); they differ in that the tabulations for the stars are entirely separated from those for integral degrees of declination, that ten degrees of latitude (instead of five) are included in each volume, and that the altitudes are not corrected for refraction. M. Jacq's note gives the basis of the tables and the method of preparation. It will suffice here therefore to mention that preliminary matter and explanations have been reduced to a minimum, and that most of the auxiliary tables have been taken directly from the ANT, the interpolation tables on the end pages being reproduced photographically. The new matter consists of star identification tables which give the declination and hour angle for every 3° of latitude and 4° of altitude and azimuth; these are clearly taken from H.O. 214.

The tables are reasonably well printed (though using modern face figures instead of the head and tail figures in the ANT), bound and indexed. No gross errors have been found, though no systematic check has been made. However, there are many end-figure errors in the star volumes due to the method of correcting the altitudes in the ANT. Examination shows that the refraction was subtracted according to a simple critical table, which could not of course agree with that used in constructing the tables; in some cases this leads to inconsistent values for the meridian altitudes in consecutive latitudes. For the integral degrees of declination, all quantities ending in 5 in H.O. 214 have been rounded-down.

Note sur les Tables de Hauteur et d'Azimut, A.A.F. Type 10, May, 1948, by M. J. Jacq.

Historique: Peu de temps avant la guerre de 1939, l'emploi de tables d'éléments précalculés avait paru très interessant à quelques services utilisateurs, pour la résolution du problème du point astronomique par la méthode de la droite de hauteur de Marcq St.-Hilaire. En particulier les *Tables of Computed Altitude and Azimuth*, publication américaine H.O. no. 214, et surtout les *Astronomical Navigation Tables* anglaises avaient retenu notre attention comme étant les plus agréables d'emploi.

Aussi, en 1942, le Service Technique Aéronautique décida l'édition de tables de navigation astronomique analogues du point de vue présentation aux ANT, mais avec quelques

modifications. A cette époque, il n'était prévu que l'édition de tables relatives aux astres de déclinaison comprise entre 29° Nord et 29° Sud, d'une part par raison d'économie, et d'autre part par ce qu'elles étaient completées par des albums de courbes d'étoiles STA dont l'édition était également entrepris. Ces albums, relatifs aux 6 étoiles suivants: Alioth, Deneb, Dubhé, la Chèvre, Schèdir, Véga, analogues aux albums américains Weems, permettent, d'une part de déterminer les coordonées géographiques de la position de l'aéronef d'après les mesures de hauteur de deux ou trois étoiles choisies, et d'autre part de déterminer la hauteur et l'azimut de l'une de ces 6 étoiles. Chaque album couvre 30 degrés de latitude.

Après la libération, l'État-Major de l'Armée de l'Air demanda l'édition complémentaire des tables relatives aux étoiles principales, tout en conservant la disposition des premières tables.

À la date du 1er janvier 1948, les volumes suivant ont été édités: Type normal: 2,3,4,5; Type étoiles: 0,1,2,3,4,5,6.

Sont en cours d'édition les volumes 0,1 et 6 du type normal (déclinaisons comprises entre + et − 29°,)

Aucune extension de ces tables n'est prévue actuellement.

Réalisation: Suivant les directives du STA les manuscrits ont été exécutés par un calculateur privé français, M. L. CHAMALEIX, de la manière suivante:

I. *Tables pour les astres de déclinaison comprise entre + et − 29°.*

D'après les données des *Tables of Computed Altitude and Azimuth*, H.O. 214 Américaines (édition 1940) auxquelles les corrections suivantes ont été apportées:[1]

(a). Hauteurs H: valeurs des tables H.O. 214 arrondies à la minute d'arc la plus voisine.

(b). Corrections d en fonction de la déclinaison: valeurs des tables H.O. 214 transformées en minutes d'arc (au lieu de 100$^{\text{èmes}}$ de degrés) et arrondies à la minute d'arc la plus voisine. Adoption de la table de correction des tables ANT.

(c). Azimut Z: valeurs des tables H.O. 214 arrondies au degré le plus voisin.

II. *Tables pour les 22 étoiles principales.*

D'après les données des Tables anglaises ANT, mais en ramenant ces données aux valeurs non corrigées de la réfraction.

Remarques sur la disposition des tables: 1. Nous avons estimé, au moment où fut prise la décision de faire éditer des tables d'éléments précalculés, qu'il était plus interessant que chaque volume couvre une bande de latitude de 10 degrés, au lieu de 5 degrés, comme les tables ANT, les voyages aériens projetés à l'époque comportant surtout des déplacements importants en latitude (possessions françaises d'Afrique). D'autre part il nous avait paru que les tables relatives aux planètes, au Soleil et aux étoiles de déclinaison comprise entre + 29 degrés et −29 degrés, suffiraient à la pratique de la navigation astronomique, étant donné que ces Tables permettent l'utilisation de la plupart des étoiles principales les plus connues et les plus souvent utilisées, à part quelques unes telles que Véga, la Chèvre, la Croix du Sud et Canopus.

Du point de vue instruction cette disposition conduisait à une seule méthode en fonction des arguments d'entrée dans les tables.

2. La correction de la réfraction atmosphérique pour une altitude de 5000 pieds, appliquée aux hauteurs données par les tables ne présente pas d'intérêt à notre avis. En effet, étant donné les altitudes de vol courantes il y a lieu malgré cette correction d'apporter une correction supplémentaire, aussi il nous a paru préférable de n'en pas tenir compte pour l'établissement des tables et de ne rien changer aux méthodes habituelles françaises d'application des diverses corrections aux hauteurs mesurées au sextant.

La correction de réfraction pour 5.000 pieds a été prise égale à la correction de réfraction pour 1500 m. calculée d'après les *Ephémérides Aéronautiques* françaises.

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[1] Au cours de l'établissement des Manuscrits il n'a été trouvé que deux erreurs grossières sur les Tables H.O. 214 et un certain nombre d'autres erreurs, mais malheureusement elles n'ont pas été relevées systématiquement.

576[U].—JAPAN, HYDROGRAPHIC OFFICE, *Publication* no. 685. *Skew Angle Celestial Navigation Method* (*A Method of Determining Latitude from Solar Observations made near Meridian Passage*). Tokyo Hydrographic Office, March, 1944, 10, ix p. 18.2 × 25.7 cm.

This small volume is intended for use in surface navigation when the altitude of the celestial body is greater than 70°. The methods offered are limited to latitudes between 35°S and 35°N, to declinations −25° to +25° and to meridian angles less than 15°. It contains five tables and four nomograms. Table 1 is the usual one for time-to-arc conversion. Tables 2 and 5 yield the corrections to be applied to observed altitudes greater than 70° for dip, refraction, semi-diameter, parallax and differences between air and water temperatures, for sun, moon and stars.

The nomograms are to be used with angular distances of the celestial body measured from the cardinal points of the horizon. Nomogram 1 allows one to determine the latitude corresponding to the dead reckoning longitude, using a single measure of the celestial body's angular distance from the south (or north) point of the horizon. Using as arguments, declination (0, ±25°), dead reckoning latitude (35°S−35°N), and hour angle (0, 60ᵐ), this nomogram provides the correction (0, 130′) to be applied to the measured angular distance to the cardinal point to obtain the meridian altitude. This can in turn be used with the declination to obtain the latitude.

Nomogram 2 permits the determination of the longitude corresponding to the dead reckoning latitude, using a single measure of the angular distance of a celestial body from the east (or west) point of the horizon. With arguments, declination (0, ±25°) and the measured angle (75°, 105°) to the east or west point of the horizon, one obtains the correction to the measured angle needed to give the local hour angle of the body and hence the longitude.

Nomogram 3 provides the difference of azimuth (0, 20°) of two positions of the same celestial body, when one uses the difference of the two times of observation (0, 60ᵐ) and the difference in the angles measured to the north (or south) point of the horizon (0, 30′).

This difference of azimuth (0, 10°) is used as one of the arguments in nomogram 4; the difference, latitude minus declination (0, 15°), is the other argument. With them, one obtains the correction (0, 13′), to be applied to the average of the two angles measured to the north (or south) point of the horizon to obtain the meridian altitude and therefrom the latitude.

Like all graphical methods, those presented in this volume are rapid but limited in accuracy. No account is taken of the motion of the vessel in the method of nomograms 3 and 4; since only a single observation is used with nomogram 1 or nomogram 2, this difficulty does not enter with them.

CHARLES H. SMILEY

577[V].—HARVARD UNIVERSITY, COMPUTATION LABORATORY, *Annals*, v. 17, *Tables for the Design of Missiles*. By the staff of the Laboratory, Professor H. H. AIKEN, technical director. Cambridge, Mass., Harvard University Press, 1948, lv, 226 p. 19.5 × 26.7 cm. Offset print. $9.00.

These tables relate primarily to the problem of designing a missile such that the longitudinal axis, about which it has circular symmetry, shall maintain a direction which at any time is nearly tangent to the path of the missile. This is necessary in the first place to minimize the aerodynamic drag of the missile, and thus to maximize its range and average ve-

locity. It is also necessary to make the trajectory of the missile accurately predictable. Finally, the stable orientation of the missile may be essential to its proper functioning, i.e., penetration, fuze actuation, etc., when it arrives at the target.

The usual criterion for the stability of a spin-stabilized projectile (fired from a rifled gun) is $A^2\omega^2 > 4BFp$, where A and B are the axial and transverse moments of inertia, ω is the rate of spin in radians, F is the drag, and p is the distance from the center of pressure back to the center of gravity. The present tables are not concerned with the drag and the center of pressure, which ultimately depend on experimental data, but do provide valuable assistance in the often tedious work of determining the moments of inertia and the center of gravity, as well as the mass of the missile.

The tables thus have a purely geometric basis, and deal with the several forms which are important in missile design, namely ogives, frustums of right circular cones, and fillets, with the greatest emphasis on the first. The distinction between the terms ogive and fillet is one of convenience rather than logic. Since it offers difficulty it will be described in some detail. Both involve the surface generated by the rotation of a circular arc about an arbitrary axis lying in the plane of the arc. The general term is ogive, but the "standard" ogive is restricted to circular arcs which are convex (as viewed from the side away from the axis), and have the axis lying between the arc and its center. In addition there is one class of non-standard convex ogives and two classes of concave ogives. All are terminated by planes normal to the axis. A convex ogive is called tangent if terminated at its maximum cross-section. Any convex ogive is expressible as the difference of two tangent ogives. A segment of a sphere is a convex ogive, the limiting case between the standard and non-standard types.

Fillets are non-standard ogives which are assumed to be small and are therefore given a different sort of treatment. There is a change of terminology in that they are called inside and outside instead of convex and concave, apparently with reference to the associated conical frustum. The generating area is a mere sliver bounded by the circular arc and the intersecting portions of the tangents to the arc at its ends. Furthermore, one of these tangents is assumed to be normal to the axis.

It is stated that the notation conforms with current practice in the Research and Development Division of the Bureau of Ordnance. We shall require the following:

l = axial length of an ogive or frustum.
R = radius of the circular arc generating an ogive.
d = maximum diameter of an ogive ($=2R \pm 2D$) or frustum.
D = distance from the center of the generating arc to the axis of an ogive.
$a = \sin\phi = l/R$.
$m = D/R$.
d_1 = diameter of smaller base of a frustum.
$x = d_1/d$.
h = axial length of fillet.
b = radial dimension of fillet measured from the (blunt) corner.
θ = angular measure of the circular arc of a fillet.

T. 1 gives the functions

$$H_1 = 20a^{-4}[2a - \sin^{-1}a - a(1-a^2)^{\frac{1}{2}} - \tfrac{1}{3}a^3],$$
$$H_3 = 24a^{-6}[a^2 - \tfrac{1}{4}a^4 - \tfrac{3}{5}[1 - (1-a^2)^{\frac{1}{2}}]],$$
$$H_5 = 28a^{-7}[\tfrac{3}{4}a^3 - \tfrac{1}{4}\sin^{-1}a + \tfrac{1}{4}a(1-a^2)^{\frac{1}{2}} - \tfrac{1}{4}a^3(1-a^2)^{\frac{1}{2}} - \tfrac{1}{6}a^5],$$
$$H_4 = 144a^{-8}[8a - \tfrac{8}{3}a^3 + \tfrac{1}{5}a^5 - \tfrac{7}{8}\sin^{-1}a - \tfrac{3}{8}a(1-a^2)^{\frac{1}{2}} + a^3(1-a^2)^{\frac{1}{2}}]$$

and their first differences to 6D for $\sin\phi = a = 0(.001).999$. These are auxiliary functions which with considerable additional calculation (see **T. 3**) yield the volume, the center of gravity, and the transverse and axial (or polar) moments of inertia of an ogive of any type.

T. 2 gives $S_0 = 3a^2H_1/20$ and its first differences to 8D for $a = 0(.001).999$. It is used in getting the volume of any type of ogive.

T. 3, which is bivariate, gives the functions

$$s = 1 - \frac{a^2}{3(1-m)} + \frac{ma^4}{20(1-m)^2} H_1$$

$$t = s^{-1}\left[1 - \frac{a^2}{2(1-m)} + \frac{ma^4}{12(1-m)^2} H_2\right],$$

$$u = s^{-1}\left[1 - \frac{2a^2}{3(1-m)} + \frac{a^4}{5(1-m)^2} + \frac{ma^4}{10(1-m)^2} H_1 - \frac{ma^6}{14(1-m)^3} H_3 + \frac{m^2a^8}{144(1-m)^4} H_4\right],$$

$$v = 4s^{-1}\left[1 - \frac{3a^2}{5(1-m)} + \frac{3ma^4}{28(1-m)^2} H_3\right] - 3t^2$$

without differences to 6D for $0 \leq m \leq .995$ and $0 \leq a < (1-m^2)^{\frac{1}{2}}$. The intervals in m and a are as follows:

Range of m	Δm	Δa
0 –.600	.020	.020
.610–.710	.010	.020
.720–.790	.010	.010
.800–.895	.005	.010
.900–.995	.005	.005

In equation (17), H_3 should read $H_3/28$, and $-a^2/(1-m)$ should read $-\frac{1}{2}a^2/(1-m)$; in equation (68a), K should read mK. These functions apply to standard ogives as follows:

V = volume = $\frac{1}{4}\pi d^2 ls$;
$\bar{x}$ = distance from base to center of gravity = $\frac{1}{2}lt$;
I_p = polar moment = $\frac{1}{8}Vd^2u$;
I'_{TCG} = transverse moment = $\frac{1}{16}Vl^2v$.

(Other variants of the transverse moment are also used in the volume.)

T. 4 gives the foregoing functions s, t, u, v without differences to 6D for $a = 0(.001).999$, and $m = (1-a^2)^{\frac{1}{2}}$. It applies to pointed standard ogives.

T. 5 gives the function $s = \frac{1}{3}(x^2 + x + 1)$ with first differences to 8D for $x = 0(.001).999$. The volume of a conical frustum is then obtainable as $V = \frac{1}{4}\pi d^2 ls$.

T. 6 gives the functions s (as in **T. 5**),

$$t = (3x^2 + 2x + 1)/2(x^2 + x + 1),$$
$$u = 3(x^4 + x^3 + x^2 + x + 1)/5(x^2 + x + 1),$$
$$v = 9[(1+x)^4 + 4x^2]/20(1 + x + x^2)^2$$

without differences to 6D for $x = 0(.01).99$. These are used in the same formulae as the corresponding functions of **T. 3**, but refer to conical frustums instead of ogives.

T. 7 gives the functions

$$s = (\tan \tfrac{1}{2}\theta - \tfrac{1}{2}\theta)/(1 - \cos\theta)(\tan \tfrac{1}{2}\theta),$$
$$t_{1A} = 1 - 1/6s, \qquad t_{2A} = (1 - 1/6s)/(1 - \cos\theta)$$

without differences to 4D for $\theta = 30°(1°)120°$. Then the area which generates the convex fillet by revolution has the measure bhs, and a center of gravity (for the area, not the solid) at a distance ht_{2A} axially and bt_{1A} radially from the "corner" of the area.

T. A, B, and **C** (in the Introduction) are bivariate tables, totalling only 24 entries each, which relate to the center of gravity and moments of inertia of convex fillets. **T. D** gives 6 values of each of three similar functions for concave fillets for which $\theta = 90°$.

The volume has an excellent introduction, with sections entitled Definitions of the Tabulated Functions, Method of Computation, Interpolation in the Tables, The Use of the Tables, and Non-Standard Ogives.

The tables were designed in collaboration with the Navy Department, Bureau of Ordnance, and the computations carried out on the Automatic Sequence Controlled Calculator, under contract with the Bureau of Ordnance.

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MATHEMATICAL TABLES—ERRATA

In this issue references have been made to Errata in "*Guide* to Tables in Elliptic Functions" (Airey, Bertrand, Dale, Dwight, FMR, Gauss, Glaisher, Gossot, Greenhill, Hancock, Hayashi, Heuman, Hippisley, Innes, Jahnke & Emde, Kaplan, Legendre, Lévy, Meissel, Merfield, Moore, Nagaoka & Sakurai, Pidduck, Plana, Potin, Rosenbach, Whitman & Moskovitz, Runkle, Samoĭlova-Ťakhontova, Schlömilch, Spenceley, Verhulst, Wayne), and in RMT **554** (Gifford), **557** (Yarden & Katz), **558** (Akushskiĭ & Ditkin, Ditkin, Lĭusternik), **568** (Legendre), **571** (De Morgan, Hammer, Soldner, Weigand), **575** (France), **577** (Harvard).

138.—E. P. ADAMS, *Smithsonian Mathematical Formulae* . . . First reprint, Washington, D. C., 1939. See also *MTAC*, v. 1, p. 191; v. 2, p. 46, 353.

P. 10, under 1.272, *for* $a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$, *read* $a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$.

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139. L. J. COMRIE, *Chambers' Four-Figure Mathematical Tables*. 1947. See *MTAC*, v. 3, p. 86–87.

The following error was found during the reading of the proofs for the new Chambers' six-figure table.

Page 64, left column, line 29. Equivalent for 1 calorie, *for* 1.363×10^{-6} K. W. H., *read* 1.163×10^{-6} K. W. H.

L. J. C.

140. G. H. HARDY & E. M. WRIGHT, *An Introduction to the Theory of Numbers*. Oxford, 1938.

On p. 71–72 the authors state that the direct converse of Fermat's theorem is false; it is not true that, if a is a prime and $a^{m-1} \equiv 1 \pmod{m}$, then m is necessarily a prime. To illustrate that the cases in which this converse is false are "rather rare," they list what they believe to be all "composite values of m below 2000 for which $2^{m-1} \equiv 1 \pmod{m}$," as "$341 = 11 \cdot 31$, $561 = 3 \cdot 11 \cdot 17$, $645 = 3 \cdot 5 \cdot 43$, $1387 = 19 \cdot 73$, $1729 = 7 \cdot 13 \cdot 19$, $1905 = 3 \cdot 5 \cdot 127$."

Another value of m, not here listed, is $1105 = 5 \cdot 13 \cdot 17$.

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141. E. JAHNKE & F. EMDE, *Tables of Functions*, 1933 (fig. 92, p. 192), all later editions (fig. 67, p. 126). See *MTAC*, v. 3, p. 41.

This figure is a relief of $J_p(x)$, for $x = 0(1)20$, $p = 0(1)10$. In my paper "Variation of bandwidth with modulation index in frequency modulation," Inst. of Radio Engineers, *Proc.*, v. 35, Oct. 1947, p. 1015, the relief of $J_p(x)$, for $x = 0(1)20$, $p = 0(1)15$ (fig. 4) shows that the Jahnke & Emde relief is inaccurate near the origin.

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142. NBSCL, *Tables of the Bessel Functions* $Y_0(x)$, $Y_1(x)$, $K_0(x)$, $K_1(x)$, $0 \leqslant x \leqslant 1$, 1948; see *MTAC*, v. 3, p. 187–188, 203.

Certain remarks in the "Foreword" of this book, on page v, are so misleading that they must be classed as erroneous.

It is stated that "In this range the British Association Tables of $K_0(x)$ and $K_1(x)$ are rather inaccurate, since both functions are singular at $x = 0$. In order to avoid the necessity for laborious interpolation in the British Association Tables when neutron distributions in graphite were needed, it was felt advisable to have a table of functions $K_0(x)$ and $K_1(x)$ for those small values of x, which, while not contained adequately in the British Tables, are very often used in neutron computations." In the first place, it is claimed on p. xi of the "Description . . ." in BAASMTC, *Mathematical Tables*, v. 6, *Bessel Functions, Part I, Functions of Orders Zero and Unity*, that the final digit in any value tabulated is within 0.52 of a unit of the true value, and this claim is still maintained by the Editor of the volume and by the Committee. Secondly, the general 8-figure accuracy of the book is maintained, and even slightly increased, even when approaching the singularity of $K_0(x)$ and $K_1(x)$ at $x = 0$. Thirdly, special auxiliary functions are given so that interpolation is readily possible, though admittedly with about twice the work, right up to the singularity.

The position is described fairly in Dr. Lowan's "Introduction," where it is stated that the values given in his tables were, in fact, derived from the British Association Tables.

J. C. P. MILLER

143. J. T. PETERS, *Zehnstellige Logarithmentafel, Erster Band, . . . Anhang*, by Peters & Stein, 1922. See *MTAC*, v. 1, p. 57–59; v. 2, p. 164.

In addition to the errors already noted in this *Anhang*, I have recently discovered that the 84D value of log 127 derived on p. XXVII, is too large by nearly 5.35×10^{-84}. My calculations of both log 127 and ln 127 were carried to 115D, and subsequently my value of ln 127 was compared with an estimate of that number to 110D, communicated to me by Professor UHLER. The agreement between the two approximations to 110D was perfect. Futhermore, the 54D value of ln M (p. 7) is correct to only 50D (*for* 63431 9772, *read* 63432 0083); and in the 61D value of log 1009 (p. 160) the 59th figure should be 2, instead of 3. [Given correctly by P. & S., p. 162, l. 1.—ED.].

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144. U. S. HYDROGRAPHIC OFFICE, *Publication* no. 214, v. 4, 1940. See *MTAC*, v. 2, p. 182–183.

In MTE 93 the statement was made that "though the sample examined is a small one, it is believed that it is fairly representative of the accuracy to be expected of H.O. 214." Information has been recently received in a letter from the U. S. Hydrographic Office indicating that v. 4 is not typical of H.O. 214. Quoting from this letter:

"It is unfortunate that most reviews of H.O. 214 have dealt with v. 4. It is true that this volume was entirely recomputed at the time the W.P.A. project was in progress at Philadelphia, working under the technical supervision of this office. Photostat copies of these

computations are available in this office. Since the volume was at first set up at the Government Printing Office, it was thought impossible to insert all the corrections on the original plates without their resulting mutilation. To replate the volume would have cost $15,000 or more. Funds available did not justify this additional expenditure. It was thought that by correcting all altitudes that were in error by more than .3′ and all azimuths in error by more than .4° the tables would be serviceable and that the remaining smaller errors, which were numerous, would result in no ill effects to the practical navigator. Errors of greater magnitude now appearing in the volume, if such exist, were apparently overlooked by the person in comparing the original printing with the later computed manuscript. Thus, the entire set of tables, excluding v. 4, are considered accurate and to contain fewer errors than most such tables, involving a multiplicity of computations and resulting figures. V. 4 is believed to be sufficiently accurate for general use.

"All latitudes were computed in duplicate by the W.P.A. project excepting latitude 20°, which had been computed and largely checked on machines in what was then the Division of Research of the Hydrographic Office. As you perhaps know the project was organized to spread work and hence the use of calculating machines was practically forbidden. The computations were made by two separate groups, the results compared, and where differences occurred the computation was redone. This explains the high degree of accuracy to be expected throughout the other 8 volumes."

In the light of this new information, it is clear that a more accurate statement would be that the Errata mentioned earlier may be considered as typical of v. 4 rather than as typical of all 9 volumes of H.O. 214.

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UNPUBLISHED MATHEMATICAL TABLES

72[F].—H. E. SALZER, *Representation Table for Squares as Sums of Four Tetrahedral Numbers.* MS in possession of the author, NBSCL.

This table shows for each square, $m^2 \leq 10^6$, a set of four tetrahedral numbers, i.e. numbers of the form $n(n+1)(n+2)/6$ whose sum is m^2. The author conjectures that every square is the sum of four such numbers. See *MTAC*, v. 1, p. 95, UMT 8.

73[K].—WILFRED JOSEPH DIXON (1915–), *Table of Normal Probabilities for Various Intervals.* Completed in 1945 at Princeton University. Manuscripts in possession of the author, University of Oregon, and of the Library at Brown University. 4 sheets of text 19.5 × 26.7 cm. 10 sheets of tables, 36 × 26.2 cm.

Prof. Dixon writes: "The table is useful for the investigation of many of the problems in order statistics. A discussion of order statistics is given in Amer. Math. Soc., *Bull.*, Jan. 1948, by S. S. WILKS. Several of the distribution functions he gives there are such that it is necessary to evaluate them numerically."

Let for $l > 0$

$$g(x, l) = (2\pi)^{-\frac{1}{2}} \int_{x-\frac{1}{2}l}^{x+\frac{1}{2}l} e^{-\frac{1}{2}t^2} dt.$$

The present tables give values to 6D of $g(x, l)$ for $x = [0(.1)5]$ and $l = [0(.1)10]$. The rows of the double-entry table correspond to fixed values of x, the columns to l. There are 49 rows and 10 columns per sheet. Let

$$\phi(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2} dt.$$

Then $g(x, l) = \phi(x + \frac{1}{2}l) - \phi(x - \frac{1}{2}l)$. The present table was computed in this way from the tables of $\phi(x)$ given by L. R. SALVOSA (*Annals Math. Statistics*, v. 1, 1930, p. 191 f.). Since the latter is only to 6D it is clear that the last digit of the present tables is not reliable.

More accurate values could be obtained from NBSCL, *Tables of Probability Functions*, v. 2, 1942. In applications one usually requires differences $\phi(b) - \phi(a)$ rather than values of $\phi(x)$. Such differences can now be obtained by a single reading, putting $x = \frac{1}{2}(b + a)$ and $l = b - a$.

W. FELLER

74[L].—HERBERT E. SALZER, *Table of* $\Gamma(n + \frac{1}{2})$, ms. in possession of the author, NBSCL, 150 Nassau St., New York City.

This is a manuscript of $\Gamma(n + \frac{1}{2})$, $n = [0(1)1000;8S$, guaranteed to 7S]. The table can also be used to obtain $\Gamma(-n + \frac{1}{2})$ by a single division, from the relation $\Gamma(-n + \frac{1}{2}) = (-1)^n\pi/\Gamma(n + \frac{1}{2})$.

H. E. SALZER

AUTOMATIC COMPUTING MACHINERY

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TECHNICAL DEVELOPMENTS

Our current contributions under this heading appear in the earlier part of this issue. They are "The memory tube and its application to electronic computation," by ANDREW V. HAEFF, and "The logical design of the Raytheon Computer," by R. M. BLOCH, R. V. D. CAMPBELL & M. ELLIS.

DISCUSSIONS

General Design Considerations for the Raytheon Computer

The basic logical units of a general-purpose automatic digital computer are: first, a memory for storing numbers and coded instructions; second, an arithmetic unit for performing addition, subtraction, multiplication, division, and other required operations; third, a central controlling facility for directing the machine in accordance with the coded instructions; and fourth, auxiliary devices required for transforming the input data into a form suitable for the machine, and for recording final results.

In most machines, there are two types of memory provided: a unit of relatively low storage capacity in which any selected item in the unit may be obtained quickly; and another unit, having a much larger capacity, but from which items can be obtained quickly only if they are used in a preassigned order.

In the design of computers, there are a number of basic decisions which together largely characterize the machine. These decisions relate to the manner of representing numbers and instructions inside the machine, the basic or built-in operations provided in the arithmetic unit, and the framework within which machine operations are scheduled.

Representation of Numbers.—In digital machines, the radices 2 and 10 are almost universally used for number representation. Normally the digits used are non-negative, except possibly during the performance of certain operations such as multiplication and division.

The use of the radix 10 has the advantages that numbers and arithmetic processes are in a familiar form. The normal form of the digits of the decimal representation is not necessarily preserved, however, since in many machines it is necessary to use a coded decimal notation. Thus, if bi-valued storage elements are used for number representation, a set of at least four such elements must be used to obtain 10 distinct configurations. The binary system makes for simpler arithmetic processes than does the decimal or coded decimal and frequently requires less equipment for the storage of numbers.

If a digital computer is used as part of a control device, it may receive its numerical input data from instrument readings. Such readings, if in continuous form, must be con-

verted to discrete form before being introduced into the machine. The conversion to binary should be no more difficult than the conversion to decimal. For control purposes it may be required to convert the output of the computer back into continuous form. Here again, using the binary system rather than the decimal within the computer creates no additional conversion problems.

By contrast, a digital computer operating from data which are provided in decimal form, and required to deliver a decimal output, must have facilities for automatic conversions in scale of notation if the binary notation is used internally. Methods of conversion will be discussed later.

In any scale of notation there are two commonly used methods for representing numbers. Let the radix used be r. Then, in the first, or fixed-point method, all numbers X in the machine are of the form $X = \frac{X}{|X|} \sum_{j=a}^{b} x_j r^j$, where a and b are constants of the machine,[1] and $0 \leq x_j \leq r - 1$. For example, in a decimal machine, a and b might assume the values -10 and -1 respectively, and all numbers in the machine would lie in the range: $-0.99999\ 99999 \leq X \leq 0.99999\ 99999$. In the second, or floating-point method, numbers are represented as $X = X_0 r^n$. Here, $r^b \leq |X_0| \leq r^{b+1} - r^a$, where a and b are integral constants, and n is an integer such that $c \leq n \leq d$, c and d being integral constants. In a decimal machine, typical values for a, b, c, and d are -10, -1, -15, and 15 respectively. In this case, $0.10000\ 00000 \leq |X_0| \leq 0.99999\ 99999$ and $10^{-16} \leq X < 10^{15}$. It should be noted that special provision must be made for representing the number zero in the floating-point arrangement.

In either of the above methods of representation, the precision of the representation is $b - a + 1$ digits or columns. Many machines are provided with equipment for operating at two different precisions—a low precision which is usually standard, and a high precision, double the low one, which is the less frequently employed mode of operation.

A negative number can be stored either as an absolute value and a minus sign, or as the complement of the absolute value. The complement of a positive number X is $Z = K - X$ where K is a constant. Two commonly used values for K are r^{b+2} and $r^{b+2} - r^a$. The use of complements may require some modification of the rules of arithmetic.

In fixed-point digital machines the use of scale factors (or, in many cases, linear transformations) is essential, and may for some problems involve many complications. The scale factors are needed to insure that all variables employ as large a fraction of the useful range of number representation as possible, without exceeding this range. If, for example, a variable X is restricted to the range $0 < X < 100$, the transformation $X' = \frac{1}{50}(X - 50)$ might be employed, such that $-1 < X' < 1$. Floating-point operation greatly reduces the need for scale factors, but complicates the operations of addition and subtraction.

The position of the radix point in the fixed-point representation is a matter of some importance since it determines what combinations of operand values are permissible for multiplication and division. If $b \leq -1$, the product of any two permissible numbers is likewise a permissible number; with $b < -1$, however, precision is needlessly lost. For $b \geq 0$, certain products are excluded. Thus, from the standpoint of multiplication alone, $b = -1$ is the best value available. In the case of division, the number of excluded cases decreases as b is increased. For $b = -1$, all cases are excluded for which the divisor is not larger than the dividend; larger values of b are thus suggested.

It should also be mentioned that even the integer 1 cannot be placed in the machine without using a scale factor if $b < 0$.[2]

Arithmetic Operations.—So far, little attention has been devoted to the decision as to which operations the arithmetic unit should be designed to handle. In this connection, a distinction must immediately be made. Some operations can be initiated by a single signal from the central control unit—these will be called built-in operations. Others will be compounded of such built-in operations. If a basic set of built-in operations is furnished, a wide class of problems can be handled by choosing a proper sequence of coded instructions.

A general-purpose automatic digital computer evidently requires some means of performing—(1) addition; (2) subtraction; (3) multiplication; (4) division; (5) extraction of

square roots; (6) calculation of algebraic and transcendental functions; (7) selection of one number from a set of numbers according to the value of a controlling argument; (8) sorting, ordering, and collating of data. Moreover, the machine must have the ability to modify instructions in accordance with partial results obtained in the computation. Finally, if the machine operates in a nondecimal scale of notation, it may also be required to perform conversions from one scale of notation to another.

The minimum list of built-in operations which could be considered reasonable for most applications[3] might be adding, complementing, shifting right, and selecting. These may be defined symbolically as: adding: $X + Y = Z$; complementing: $K - X = Z$, where K is a constant; shifting right: $X \cdot r^{-1} = Z$, where r is the radix; selecting: if $X \geq 0$, select Y, if $X < 0$, select 0—call the result Z. Multiplication and division, as well as all other operations, would be compounded from the above four. For convenience in preparing problems for a general-purpose machine, however, a much larger list of basic operations should be provided.

Regardless of which operations are selected as built-in operations, the four operations listed above are usually the basis of the design of the arithmetic unit. Thus, subtraction is usually performed by adding the complement of the subtrahend to the minuend; multiplication—particularly in a binary machine—is simply a combination of additions, shifts, and selections.

Division, the extraction of square roots, and the calculation of simple functions can all be performed in a number of different ways in terms of the above operations. Division, for example, can be performed by subtraction, selection, and shifting. Or the reciprocal of the divisor can be computed by an iterative process. If X is the number whose reciprocal Y is desired, and Y_0 is a first approximation to Y such that $0 < |Y_0| < 2|Y|$, and if Y_0 and Y are of the same sign, then the iteration of the process $Y_{n+1} = Y_n(2 - XY_n)$ converges to Y.

If floating-point operation is included, either as a built-in or as a compounded operation, additional features are required in the arithmetic unit. This is also the case if numbers of twice normal precision are to be handled.

If conversion in scale of notation is required as a machine operation, it can be done by several methods. Consider the conversion of a number from the scale of notation with radix r to that with radix s. In the first scale of notation write the number as

$$X = \frac{X}{|X|} \sum_{n=n_1}^{n_2} x_n r^n (n_2 > n_1)$$

and in the second, as

$$Y = \frac{Y}{|Y|} \sum_{m=m_1}^{m_2} y_m s^m (m_2 > m_1).$$

The additions and other computations required in the conversion can be performed in either scale of notation. If they are performed in the r notation, the fractional part of X may be converted by successive multiplications by s expressed in the r notation. After each multiplication, the integral part is equal to one of the required digits y_m of Y; this integral part is truncated before the next multiplication. The integral part of X is converted by a method which is similar to the above but employs divisions by s in place of multiplications by s.

In order to perform the calculations in the s notation, one uses a slightly different method. One converts the digit of lowest order in the fractional part of X—viz., the digit for $n = n_1$—to the s notation, and then multiplies by $1/r$ expressed in the s notation. Next one adds the equivalent of the digit of second lowest order, and again multiplies by $1/r$. The alternate addition and multiplication continues until all of the digits of the fractional part of X have been used. The integral part of X is converted by alternate addition and multiplication by r. Here one starts with the digit of highest order.

Alternative conversion processes are available which make use of either the s-equivalents of $x_n r^n$ or of the r-equivalents of $y_m s^m$. The equivalents, in this case, are supplied to the calculator as part of the input data for each problem.

Representation of Orders.—Not only input numbers, but also coded instructions must ordinarily be introduced into the computer at the start of the problem. When superimposed upon the logical structure of the machine, these instructions must complete the formulation of the problem.

Consider the computation of $Z = (AY + B)X + C$, in which the numbers A, B, C, X, and Y are stored in the memory, and the result Z of this composite operation is to be sent to the memory. The computation requires the locating of the five input numbers in the memory and their transfer to the arithmetic unit, where operations are performed upon them. It also requires the specification of two multiplications and two additions. The quantity Z must be sent to the proper memory location, since that location will serve to identify Z for future use. Finally, all of the foregoing processes must be performed in the correct sequence. The performance of these multiplications and additions, as well as number transfers between the memory and the arithmetic unit, requires the use of certain coded instructions which must be selected from the memory in the proper order and must then be used to activate the necessary controls.

In any computing process, the required operations are usually performed as a succession of rather simple steps. The entire amount of instructional data needed to formulate each step is selected from the memory as a single unit, which is called an order. A number of different schemes can be used for the breakdown of the computation into steps, and the corresponding breakdown of the instructional data into orders. If the basic step consists of a transfer between the memory and the arithmetic unit, together with the possible initiation of an arithmetic operation, the computation of Z above requires six steps: five to send the input data to the arithmetic unit, and one to return the result to the memory. Another type of basic step would include a complete multiplication or addition and the necessary number transfers. Then the computation of Z would consist of four steps. Still other basic steps could be constructed involving, for example, a pair of number transfers, or even a pair of arithmetic operations, such as an addition and a multiplication. The various choices of basic steps will evidently correspond to different types of orders.

Orders as well as numbers must be selected from the memory. Orders are then sent to the central controlling unit of the computer. The sequence in which the orders are chosen from the memory can either be preassigned as part of the logical structure of the machine, or it can be specified by the orders themselves. If a preassigned sequence is used, provision must be made for a departure from this sequence under special circumstances. The storage of orders in the internal memory generally permits the use of either method of order selection. Order storage in the external memory, however, requires that the orders be used in a preassigned sequence—namely, that in which they are recorded on the external storage medium.

Orders must contain at least two types of coded instructions—an instruction governing the transfer of a number between the memory and the arithmetic unit, and an instruction initiating one of the built-in operations of the arithmetic unit. To specify a transfer, one must locate each terminus and state whether the number is to enter or leave the memory. All of this information need not be explicit in the order, however. The specification of a location may be called an *address code*. The codes for these locations in the internal memory are conveniently assigned as consecutive integers expressed in the same scale of notation as that used for the actual numerical quantities. Data are normally taken from or routed to the external memory sequentially; hence, the address code of a given location need only specify which of several memory units is involved. Special instructions for nonsequential use of the external memory should also be included.

Since it must be possible to modify orders in accordance with partial results obtained, it is desirable to handle orders in the arithmetic unit in the same manner as numbers; this, in turn, requires certain similarities in their respective modes of representation.

The simplest type of machine organization is based upon completely sequential operation. The several processes controlled by each order take place in succession, and no two orders are operative simultaneously. Such serial operation uses a minimum of equipment and simplifies the operation and testing of the machine.

Higher speeds can be obtained through the scheduling of two or more operations simultaneously. Thus, for example, two transfers, or a transfer and an arithmetic operation, can be scheduled simultaneously. Composite computers, containing two or more arithmetic organs, enable several arithmetic processes to be performed concurrently.

Another type of choice is available in the scheduling of machine processes. Each basic step can be allocated either a length of time which varies with the nature of the step, or one which is fixed. The latter mode may simplify the controls for certain types of machine construction but is somewhat wasteful of operating time, since all steps must be allowed the maximum length of time whether it is needed or not.

A sample logical design will illustrate the nature of the alternatives just discussed. Each order will be taken to specify one number transfer and, when required, the initiation of an arithmetic process. Successive orders are normally taken from successive locations in the memory; hence, no explicit instructions for the locations of orders need be given, except in special cases. Each order consists of a memory address code and an operation code. The latter, taken in conjunction with the address, completes the specification of the transfer. Using this type of order, the sample calculation $Z = (AY + B)X + C$ could be represented by the following order sequence:

Number of Order	Address	Operation Code
1	Address of A	Transfer to arithmetic unit.
2	Address of Y	Transfer to arithmetic unit; multiply by A.
3	Address of B	Transfer to arithmetic unit; add to above product.
4	Address of X	Transfer to arithmetic unit; multiply by above sum.
5	Address of C	Transfer to arithmetic unit; add to above product.
6	Address to which Z is to be delivered	Transfer above result from arithmetic unit to memory.

It is assumed in the above sequence that the arithmetic unit is capable of remembering partial results (e.g. the product $A \cdot Y$) between arithmetic operations.

Checking and Errors.—Four kinds of errors (apart from human errors) may occur in digital computation: 1. Errors due to inaccuracies in the numerical input data; 2. Round-off errors occurring in multiplication and division, and possibly also in other operations; 3. Errors due to the inadequacies of the computing procedure (often called truncation errors); 4. Errors due to malfunctionings of the machine.

Only errors of the fourth type will be considered here, and the term error will be restricted to cover this case only.

If a computer is to be of value, its error frequency must be sufficiently low, and the average time to diagnose and repair troubles sufficiently short, to enable the machine to be operated successfully a large fraction of the time. When the computer does make an error, it is essential that the error be detected, so that no wrong results are obtained. A low error frequency and a short repair time can be achieved by having a minimum amount of equipment, and by careful engineering and packaging. The detection of errors (i.e. checking) and error diagnosis can be conveniently considered together.

Two general types of checking may be distinguished: first, checking which requires changes in the computing routine but practically no increase in machine equipment; and second, checking which depends primarily upon the addition of extra equipment. It should be observed that the use of either type of checking tends to cause an increase in the number of errors per problem.

In applications where machine stoppage is extremely serious, even if the failure is detected and diagnosed, the use of extra equipment or extra computation for checking purposes may not be desirable unless it is essential to know whether a failure has occurred. If, however, the error frequency can be kept to a reasonably low figure, and a temporary shut-down of the equipment is not too serious, checking is ordinarily very worthwhile. Mathematical checks (i.e., those involving only an increase in computing time) seem to be of

somewhat limited application. If the mathematical formulation of the problem is made self-checking, little diagnosis is ordinarily obtained, and in many cases the volume of the computations is doubled. Test problems can be made diagnostic, but unfortunately these are not necessarily adequate checks; all of the conditions relevant to the behavior of the machine on the main problem are not necessarily reproduced in the running of the test problem. Thus, if both checking and diagnosis are desired, there appears to be a distinct advantage in building the checks into the equipment.

In the extreme case where it is desired to have the machine not only locate its own errors but also correct them and continue automatically, a rather considerable amount of extra equipment may be required. On the other hand, a reasonable amount of checking and diagnosis, without the feature of self-correction, need not involve large quantities of equipment.

In a self-checking machine each number-word must evidently contain information not only as to the value of the number represented, but also as to its correctness. The set of all possible words contains a subset of valid words—i.e., words in which the criterion for correctness is satisfied; all other possible words are invalid. An elementary example of a self-checking method of number representation involves using complements. Each valid number-word consists of an n-digit binary number, together with its n-digit complement. If the second number is not the complement of the first, the word is invalid. There are thus 2^{2n} possible words, of which 2^n are valid.

A method of checking the storage and transfer of words has been devised which detects a large fraction of the machine errors but does not require much extra storage capacity. Let an unchecked number X of 35 binary columns be expressed as

$$X = \frac{X}{|X|} \sum_{n=0}^{34} x_n 2^{n-35},$$

where x_n is either zero or unity. Consider the auxiliary number formed by taking the weighted sum of the digits[4] x_n of X, viz.,

$$X_c = \sum_{n=0}^{34} x_n w_n 2^{-35}.$$

The auxiliary number X_c is stored with the original number X to form the complete number-word and will be termed the weighted count of X. The validity of a word after storage and transfer is checked by obtaining a new weighted count and comparing it with the one obtained previously and stored in the word. The weighted count is most conveniently calculated using weights which are all powers of 2, although not necessarily the same power.

By a judicious selection of the weights w_n, an arithmetic check can also be obtained. Let the w_n be 1, 2, 4, 8, 16, 1, 2, 4, 8, 16, 1, 2, 4, 8, 16, $\cdots$ for $n = 0, 1, 2, 3, \cdots$ respectively. Thus the weights repeat in groups of five, and within each group are just the successive powers of two. It can easily be seen that the weighted count X_c is merely $\sum_{m=1}^{7} y_m 2^{-35}$, where the quantities y_m are the digits of X when X is written in the scale of notation with radix 32. It therefore follows that $X - X_c = 31K \cdot 2^{-35}$, where K is a non-negative integer. Thus, the weighted count of the sum of two numbers differs from the sum of their weighted counts by a multiple of 31. This fact can be used to check the addition process in the arithmetic unit; the other arithmetic operations can also be checked by means of the weighted count.

The checking of selection can be accomplished briefly as follows. The machine is so arranged that whenever one device in a group of N is selected, that device sends back a tag or identification signal. The tag can be compared with the portion of the instruction which governed the original selecting process, and any discrepancy will indicate a false selection.

R. M. Bloch, R. V. D. Campbell & M. Ellis

[1] In some machines, provision is made for changing the parameters a and b between problems.

[2] See S. LUBKIN, "Decimal point location in computing machines," *MTAC*, v. 3, p. 44-50.—EDITOR.

[3] It is here assumed that to build up all processes from the basic operations of logic would require far too many orders in the formulation of problems.

[4] The algebraic sign of the number is also taken into consideration in forming the weighted sum X_s; this has not been indicated in the formula, however.

BIBLIOGRAPHY Z–V

1. ARTHUR W. BURKS, "Electronic computing circuits of the ENIAC," Inst. Radio Engineers, *Proc.*, v. 35, 1947, p. 756–767; illustr. 27.9 × 21.6 cm.

The ENIAC "is a very large device (containing 18,000 vacuum tubes) compounded out of a few basic types of computing circuits. The design principles that were followed in order to insure reliable operation of the electronic computer are presented, and the basic types of computing circuits are analyzed.

"Most of the design work on component circuits was devoted to constructing reliable memory circuits (flip-flops) and adding circuits (counters). These are treated in detail.

"The ENIAC performs the operations of addition, subtraction, multiplication, division, square-rooting, and the looking up of function values automatically. The units which perform these operations, the units which take numerical data into and out of the machine, and those which control the over-all operation are described.

"The technique of combining the basic electronic circuits to perform these functions is illustrated by three typical computing circuits: the addition circuit, a programming circuit, and the multiplication circuit."

2. GREAT BRITAIN, POST OFFICE ENGIN. DEPT., Research Report no. 12719: A. W. M. COOMBS & W. W. CHANDLER, *Automatic Computing: An Analysis of Arithmetical Operations*. Post Office Research Station, Dollis Hill, London, N. W. 2, Aug. 1946. 29 p. + 1 plate. Mimeographed. 22.2 × 27.9 cm.

3. STIG DJURE & EINAR WELIN, "L. M. Ericssons Nya Automatiska Kalkylator for Regleringsändamål," [L. M. Ericsson's New Automatic Calculator for regulating outputs], *Teknisk Tidskrift*, 1944, fasc. 48, 32 leaves, 55 figures; French resumé. 20.3 × 29.2 cm.

Given a certain number of functions of time, $a(t)$, $b(t)$, $c(t)$, the computer under consideration yields the values of certain functions $f_1(a, b, c \cdots)$, $f_2(a, b, c \cdots)$, etc. when t varies continually, and without appreciable time loss. The analytical expressions of the "f" functions can contain derivatives, integrals, elementary functions; they can also be given as implicit functions of $a, b, c, \cdots$ or as the solutions of a system of equations. An example of a system treated on a machine of this type is presented.

It must be understood that a machine adjusted to solve a certain system of equations will solve only this system. The possible standardization of the machine elements will, however, make it adaptable to other similar systems.

4. ADELE K. GOLDSTINE, *Report on the ENIAC* (developed under the supervision of the Ordnance Department, United States Army), University of Pennsylvania, Moore School of Electrical Engineering. *Technical Report* I, bound in 2 v. (v. 1—chaps. I–VI; v. 2, chaps. VII–XI) June 1, 1946, xxxvi, 301 p.; 122 figs. Mimeographed. 21.6 × 27.9 cm. See *MTAC*, v. 2, p. 97–110, an article by H. H. GOLDSTINE & A. K. GOLDSTINE.

Without going into the details of the circuits, this Report covers the functions of the various units of the ENIAC and their integration. It thus serves as the logical introduction

to the machine. Those who are not interested in circuit details nor in maintaining an electronic computer need only read Technical Report I for a complete understanding of the logical design of the ENIAC. The programming of the machine is quite thoroughly covered with several illustrative problem setups.

The 6 chapters in v. 1 cover, besides an introduction, the initiating unit, the cycling unit, the accumulator, the multiplier, and the divider and square root unit. A brief general program control with relation to the various units is discussed concurrently with the units themselves. The organization of the material, therefore, differs from the more recent discussions of computers, such as the one by BURKS, GOLDSTINE, & VON NEUMANN, *Preliminary Discussion of the Logical Design of an Electronic Computing Instrument* (see *MTAC*, v. 3, p. 50–53, 128). In this latter the major divisions are input, output, memory, arithmetic unit, and control. The ENIAC itself is not "organized" into these units. It is probable that the logical separation of a computer into these five organs was not made until the ENIAC was nearly complete.

Numbers are represented in the ENIAC by parallel trains of pulses. Ten decimal digits, the normal precision, are represented by trains of from 0 to 9 pulses along 10 wires. An eleventh wire carries either no pulses, representing a positive sign, or nine pulses indicating that the ten digits represent a complement with respect to 10^{10}. Thus, to subtract X from y, $10^{10} - X$ is added to y. The 9 on the sign wire is more properly called a complement indicator. The pulses in the unit train "step" a 10-stage ring counter. When the unit counter steps from 9 to 0, a carry-over to the 10's counter takes place. The complement procedure eliminates the necessity for these counters to step backwards as well as forwards.

There are 20 accumulators, which provide high-speed storage as well as facility for addition and subtraction. Each accumulator has five numerical input channels through any one of which it can receive a 10-digit signed number. There is also provision for shifting so that an accumulator can receive a number which has been shifted a fixed number of places to the left or right. Since repeat controls are provided for additions involving the numbers 1 to 9, multiplication by a constant can be effected without the use of the high-speed multiplier.

The sequencing of the ENIAC is controlled both locally and centrally. All the units of the machine, except parts of the input and output, operate synchronously, based upon a cycle of 20 pulses of the central clock, which emits pulses at the rate of 100,000 pulses per second. A program pulse must stimulate an accumulator to transmit or receive. When an operation is complete, always an integral number of cycles after starting, a program pulse is emitted, causing the next operation to begin. Thus, a sequence of operations can be carried out by plugging together various units and associated program controls. Since there are multiple channels, several of these sequences can be carried out simultaneously. These program sequences are tied together through the use of the master programmer. Hence, the final program pulse of a sequence may be delivered to the master programmer, and another sequence can pick up its initial pulse from the appropriate output of the master programmer. Sign or digit pulses can be used to stimulate program controls. This means that the machine can make a decision between two sequences based on earlier computation. This ability to link program sequences by discrimination makes the ENIAC control system extremely flexible.

The high-speed multiplier uses two of the accumulators so that the complete product of two 10-digit numbers can be obtained. In order to obtain speed, a one-digit multiplication table is provided. That is, if a 7 comes in on the unit line from one number, and an 8 on the unit line from the second number, a 6 is transmitted out on the unit line and a 5 on the 10's line.

The ENIAC is provided with a division unit, which operates in conjunction with a group of accumulators to hold the divisor, dividend, and quotient and do the necessary shifting. Division is accomplished by successive subtraction and testing for over-draft. The same unit can also extract the square root of a number. The square-root procedure utilizes the relation: $n^2 = \sum_{i=1}^{n} (2i - 1)$.

This volume is interesting reading for those concerned with large-scale electronic digital computers, even though several features of the ENIAC have been discarded in designing some of the computers yet to be constructed. However, the technical discussion is not recommended to the casual reader because of its detailed treatment of all units. For a summary treatment the reader is referred to, among others, an article by Professor D. R. HARTREE, in *Nature*, Oct. 12, 1946 (see *MTAC*, v. 2, p. 222; also p. 97–110).

V. 2 of the report contains detailed operating instructions for several units not discussed in v. 1. The contents of each chapter are briefly reviewed below.

VII. Function Table.—The ENIAC contains 3 function-table units, each of which can be used to store 104 12-digit functional entries against a normalized independent variable or location parameter. The stored information is entered manually on switches and can be made available to other units of the ENIAC in as little as 5 addition times (1 millisecond). A single reference to the table may be made to yield not only the tabular value corresponding to the argument specified but also the 4 neighboring entries as well. This characteristic facilitates the use of polynomial interpolation formulae of fourth degree or less. The author gives explicit setting-up instructions for carrying out quadratic and biquadratic Lagrangian interpolation. Higher-degree approximations are practicable but require more elaborate programming.

The storage of ballistic drag functions in the Function Table is described in some detail. Various artifices must be employed to circumvent difficulties which arise on account of the anomalous behavior of such functions in the neighborhood of sound.

The possibility of storing programming instructions in the Function Table is touched upon only briefly in this report. Experience subsequent to the publication date (June 1946), however, has shown that storage of instructions in the Function Table can achieve considerable savings in machine set-up time and labor, although at the cost of somewhat slower machine operation.

VIII. Constant Transmitter.—The Constant Transmitter unit, operating in conjunction with an IBM card reader, provides a special auxiliary memory for the ENIAC. As many as 80 digits may be read into the Constant Transmitter from a standard IBM punch card in about one-half second. Once stored in the Constant Transmitter, these data (in addition to 20 digits and 4 signs manually set up on switches) may be referred to repeatedly by other units of the machine, the access time in each case being 1 addition time (200 microseconds). Extensive set-up details are given for utilizing this valuable feature of the machine.

IX. Printer and IBM Gang Punch.—The output equipment associated with the ENIAC consists of modified IBM units. Detailed operating instructions for plug-board wiring, switch setting, etc., are provided.

X. Master Programmer.—The over-all direction of the ENIAC's operations is under the control of the Master Programmer. This central control unit supervises the progress of the computing program and issues instruction signals which stimulate the various subsidiary organs to perform their specialized subroutines. For example, the Master Programmer is needed for controlling the iteration of a sequence of operations constituting a program chain and for linking various chains together to form subprograms of higher order. The Master Programmer can be made to link programs together either in some preassigned order or on the basis of magnitude or sign discrimination. As a typical example of the latter process, the writer describes a step-wise integration trajectory computation. In this case detailed printed results are desired only at the summit of the trajectory and at the ground. At each step the shell's altitude coordinate and its first (time) derivative are tested to see whether they lie inside a certain range close to 0. If either one is found to fall within specified critical limits, the Master Programmer is made to issue signals which cause the program to enter into the appropriate new phase.

XI. Transmission Systems.—In this final chapter is a discussion of the three principal types of dynamic communication between units of the ENIAC. These are (1) communication by means of synchronizing signals; (2) transmission of numerical data; (3) transmission of program instructions. Detailed charts and block diagrams are provided to illustrate various typical and special interconnection schemes.

General Remarks.—V. 2 is essentially an operating instruction manual for setting up those phases of the ENIAC's problems which involve the units discussed above. The emphasis throughout the volume is on logical relationships, not electronic circuitry. In fact, little or no acquaintance with electronic nomenclature is required on the part of the reader. Because of the vast wealth of detail presented, however, it seems unlikely that the report will attract many thorough readers outside of the group actually engaged in performing manipulations on the ENIAC. Nevertheless, the reader with less specialized interest who is familiar with desk computing machinery should find even a cursory inspection of v. 2 quite rewarding. For one thing, it will convince him that not even the superhuman speed and versatility of the ENIAC can relieve the human operator of the burden of tedious manual routine ordinarily involved in carrying out an extended program of numerical calculation. A brief examination of the report will make it clear that the operator is able to convert the machine to his purpose only at the cost of much tedious preliminary labor. The prodigious speed and industry of the ENIAC, however, repay this initial expenditure many thousandfold in terms of desk-machine output. Nevertheless, it is encouraging to know that the newest electronic descendants of the ENIAC will be able to achieve at least equivalent results with only a tiny fraction of the effort and attention that the ENIAC currently demands of its keepers.

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EDITORIAL NOTE: Although these v. contain many figures, tables, and drawings, reference is made to several drawings which are not included. Hence it is preferable to have the complete file of ENIAC drawings available while studying the text.

5. HARRY D. HUSKEY, *Report on the ENIAC* (developed under the supervision of the Ordnance Department, United States), University of Pennsylvania, Moore School of Electrical Engineering. *Technical Report* II, June 1, 1946, 163 p.; 12 figs. Mimeographed. 21.6 × 27.9 cm. Restricted (not for distribution).

This volume is the fifth of a 5-v. *Report on the ENIAC* (see no. 4, for v. 1–2). It is a step-by-step description of the operation of the circuits of the ENIAC written to accompany the block diagrams and circuit diagrams. These functional circuit descriptions prepare one for the *ENIAC Maintenance Manual* (v. 2), but they do not analyze the factors affecting decisions in engineering design. Voltages, tube types, and component values are given; but currents, powers, stray capacitances, rise times, wave forms, etc., are not. Absorbing the contents of this volume should be a very time-consuming task, worthwhile only for one who is to service the ENIAC.

ROBERT D. ELBOURN

NBS

6. IBM, (a) *IBM Selective Sequence Electronic Calculator*. 16 p. printed brochure, p. 5–6 = 6 p. paster; illustr. 22.9 × 30.5 cm.; (b) "IBM selective sequence electronic calculator is dedicated to science," *Business Machines*, v. 30, no. 11, 15 Mar. 1948, p. 1–12; illustr. 45.7 × 58.4 cm.

NEWS

American Physical Society.—On Friday morning, April 30, at the NBS, members attending the Washington meeting heard Mr. H. ZAGOR of the Reeves Instrument Corporation speak on the REAC. This machine is being developed under contract with the Office of Naval Research.

Mr. Zagor described a Reeves Electronic Analog Computer (REAC) employing direct-current computing elements and servos for the solution of initial-valued nonlinear total differential equations. Solutions of some differential equations were shown, as well as several interesting slides demonstrating the machine components.

The basic component of the REAC is a direct-current amplifier, of which each standard machine has a total of 20, consisting of 7 integrating amplifiers, 7 inverting amplifiers, and 6 summing amplifiers. The servos make it possible to carry out multiplications of variable quantities, to perform resolution of vectors, and to introduce arbitrary nonlinear functions. Thus, each standard REAC can solve differential equations up to the seventh order. However, several REAC's can be connected together to increase the capacity of the equipment where higher-order problems are involved. Solutions are recorded on a six-channel recorder providing graphical presentation of the effect of changing one or all of the parameters of the problem being solved. Insertion of problems is carried out by plugging of telephone cords to bays on the front of the cabinet so as to connect the input and output circuits of each amplifier in any desired manner. In order to permit continuously adjustable coefficients, 24 micropots with micrometer dials are provided. The over-all accuracy of the computer is limited by the resistors, capacitors, and drift in the power supplies. A REAC prototype has been in operation for over a year on Navy problems.

On Saturday afternoon, May 1, at the Department of Commerce Auditorium, the following papers on computing machines were presented:

"Computing machines" by Professor HOWARD AIKEN, Harvard University.

"Status of large-scale computing devices in the United States" by Mr. R. L. SNYDER, JR., University of Pennsylvania.

"The IBM selective sequence electronic calculator" by Dr. W. J. ECKERT, IBM.

International Scientific Radio Union (American Section) & Institute of Radio Engineers.—At this joint meeting in Washington, D. C., May 5, 1948, a talk on "Basic digital arithmetic circuits" was delivered by Mr. T. KITE SHARPLESS, Technitrol Engineering Co., Philadelphia, Pa. Mr. Sharpless presented some basic digital adding circuits in block and schematic diagram form. Both serial (time division) and parallel types were included. Adders using binary base and decimal base arithmetic were treated, with some discussion of the advantages and disadvantages of each. He also took up methods and circuits for taking care of subtraction or the algebraic addition of signed numbers. Two multiplying circuits, the repeated addition type for binary numbers, and a multiplication table type for decimal numbers, were presented and discussed. The problem of division was covered, but no specific division circuits were shown, since this operation can be achieved rather easily by combinations of the other basic operations. All the above-mentioned circuits use standard electronic tubes and circuit components. However, the paper concluded with a brief presentation of some special types of electronic tubes suitable for use as adders.

NBSMDL.—Each week members of the staff of the MDL at the NBS meet with other interested persons to discuss various existing and projected electronic computers.

On March 11, 1948, Mr. R. R. SEEBER, JR., of the IBM gave a detailed description of the recently completed IBM Selective Sequence Electronic Calculator. With the aid of slides Mr. Seeber explained each part of the machine and outlined the role it plays in the performance of a computation. The discussion was of interest not only to those who had not yet seen the new machine, but also to the many present whose brief acquaintance with it at the opening ceremonies left them with an incomplete knowledge of its intricacies.

On April 13 the group heard Dr. SAMUEL LUBKIN speak on the proposed Reeves Arithmetic Computer (REEVAC). A description was given of the various features and expected performances of the machine, which has not yet been constructed. Since most of those present were acquainted with comparable features of the two electronic computers designed under the direction of the NBS by the Raytheon Manufacturing Company and the Eckert-

Mauchly Computer Corp., the talk was followed by a discussion of the relative merits of the three models.

Dr. HARRY D. HUSKEY of the MDL gave two talks at Ohio State University on May 24 at the invitation of the Graduate Mathematics Club and local chapter of Pi Mu Epsilon. In the afternoon talk, entitled "Mathematical aspects of electronic computing machines," newly planned high-speed digital computers were briefly described; the responsibilities of operating personnel were mentioned; and the impact of such machines on applied mathematics and engineering was considered. The evening talk, "Engineering aspects of electronic digital computers," described typical computer components and gave a brief indication of probable future trends of development.

On June 1 Dr. Huskey gave a one-hour invited address on "The present state of automatic digital computing machinery" at the semiannual meeting of the Association of Mechanical Engineers in Milwaukee. The history of the development of digital computing machines was covered as an introduction to the subject, and the various large-scale digital computing machines were discussed. The proposed new machines which are just now in the manufacturing stage were described in some detail, with special mention of their implications in the fields of applied mathematics and engineering and with particular emphasis on points of difference from the older machines. The generality of the new machines was stressed, and the interpretation as a general model was explained, along with some of the difficulties which must be overcome before this ideal of a general model can be realized.

As a sequel to Mr. Seeber's survey of the IBM Selective Sequence Electronic Computer, on June 15, at the NBS, Mr. KENNETH CLARK, also of the IBM, presented an informative treatment of the coding of an actual problem. In the course of this discussion many essential machine features not previously mentioned were brought out.

Personal.—STIG EKELOF, professor of Electro-Technique at Chalmers University, Göteborg, has recently returned from a trip to the United States, where he reviewed the digital computer program. Professor Ekelof is constructing a mechanical analog computer (differential analyzer).

OTHER AIDS TO COMPUTATION

Electric Root-finder

The analogy between the functions of a complex variable and the flow of electric currents in a conducting sheet has been used extensively to solve problems about currents. The inverse procedure does not seem to have been exploited as profitably as it might.

One useful function of a complex variable is the characteristic determinant whose roots are the natural frequencies of a dynamical system. Unfortunately, it is a tedious calculation to find these roots by any numerical process, when the degree of the determinant is high. An electric root-finder based on the current-flow in a conducting sheet was devised by F. LUCAS,[1] 1888–1890, but it suffers from two practical defects. First, the sheet must theoretically be infinite, which means in practice that it must be very large; and second, the current density at certain current sources must be large.

These defects can be avoided by a conformal transformation of the plane which carries the upper half-plane into a bounded region, but expands the areas surrounding the current sources.

To find the roots of various polynomials in z, we select a set of real values of z, say z_1, z_2 etc., according to any convenient scheme. The number of these values must be greater than the highest degree of the polynomials to be

solved. Now form the polynomial

$$Q(z) = (z - z_1)(z - z_2) \cdots (z - z_n).$$

If $P(z)$ is a given polynomial, find the residues of $P(z)/Q(z)$ at $z_1, z_2, \cdots z_n$.

By Lucas's scheme, the upper half of the complex z-plane is plotted on a uniform conducting sheet, that in theory is semi-infinite. Currents proportional to the residues of P/Q are fed into the sheet at the corresponding points in the z-plane. Then the components of the current density at any point in the sheet are proportional to the real and imaginary parts of the function on P/Q. In particular, at a root of $P(z)$, the current density is zero. To find the roots of $P(z)$, therefore, we need only search out the points of null current and read the value of z at these points.

The same relations hold in a w-plane, where w is any analytic function of z. A useful transformation is obtained by writing

$$w = re^{i\theta},$$

where

$$\theta = 2 \tan^{-1} z, \qquad r = 1 + K/(1 - \cos n\theta).$$

This carries the real axis of the z-plane into a symmetric figure with n radial spike-like projections, whose widths are determined by the value of the real constant K.

In particular, if we choose the poles z_1, z_2 etc. at $z = \tan(2\pi/n)$, the poles of the w-plane are at the tips of the spikes. Now the length of any line segment in the w-plane is related to the length of the corresponding line segment in the z-plane by the ratio $|dw/dz|$ which is very large near one of the points $z_1, z_2, \cdots z_n$. Since the same current flows across both line segments, the current density in these regions of the w-plane is reduced, and in fact becomes finite instead of infinite as we approach the poles. Of course this is advantageous in an electrical system, where the energy dissipation must be considered.

G. R. STIBITZ

[1] See *MTAC*, v. 1, p. 346–348, 352–353. F. LUCAS, Acad. d. Sci., Paris, *Comptes Rendus*, (a) "Généralisation du théorème de Rolle," v. 106, 1888, p. 121–122; (b) "Détermination électrique des racines réelles et imaginaires de la dérivée d'un polynôme quelconque," *l. c.*, p. 195–197; (c) "Résolution électrique des équations algébriques," *l. c.*, p. 268–270; (d) "Détermination électrique des lignes isodynamiques d'un polynôme quelconque," *l. c.*, p. 587–589; (e) "Résolution immédiate des équations au moyen de l'électricité," *l. c.*, p. 645–648; (f) "Résolution des équations par l'électricité," *l. c.*, p. 1072–1074; (g) "Résolution électromagnétique des équations," v. 111, 1890, p. 965–967.

BIBLIOGRAPHY Z–V

1. ALFRED B. BABCOCK, "Slide rule for rapid computation of volumes of cylindrical, conical and rectangular tanks," *Chemical Engin.*, v. 55, no. 3, May 1948, p. 142–143. 20.8 × 28.6 cm.

2. EDWIN A. GOLDBERG & GEORGE W. BROWN, "An electronic simultaneous equation solver," *Jn. Appl. Physics*, v. 19, 1948, p. 339–345. 20 × 26.5 cm.

Summary: "An analogy device for directly solving simultaneous linear equations (without resorting to successive approximation methods) is described. It utilizes a number of high

gain amplifiers interconnected by networks whose elements bear definite relationships to the known coefficients and constants of the system of equations. The stability criterion for such a system, and its application to an equation solver are described. Since the device produces the answers to a system of equations without delay, it is adaptable to problems of synthesis as well as those of analysis."

3. FRANCIS J. MURRAY, *The Theory of Mathematical Machines*. New York, Kings Crown Press, [second] revised ed., 1948. viii, 139 p. 21.5 × 28 cm. Lithographed; plastic binding. $3.00.

The general character and contents of this interesting work is indicated in the review of its first (1947) edition; see *MTAC*, v. 2, p. 317–318. Besides minor corrections, the revision of the first edition consists mainly in adding: (A) a detailed account of trigger circuits with a good mathematical treatment; (B) a chapter on electronic digital computers with special reference to the ENIAC, the IBM electronic computer, and the proposed IAS machine; (C) a short chapter entitled "Noise, accuracy and stability"; and (D) a much needed index. The pagination has been altered so that the numbering commences anew with the beginning of each of the four parts.

D. H. L.

4. F. M. TILLER, "Stagewise operations. Graphical solutions of difference equations," *Chemical Engin. Progress*, v. 44, 1948, p. 299–306. 21 × 28.7 cm.

Summary: "Operations of multiple contact extraction, plate fractionation, and plate gas absorption are stagewise by nature. The writing of material balances for these processes leads to difference equations just as material balances for differential processes lead to differential equations. In this paper a summary of graphical methods for a number of different types of equations is given. The methods are illustrated with examples in binary fractionation, plate stripping and stagewise chemical reactions." Part I of this paper, "Stagewise operations—applications of the calculus of finite differences to chemical engineering," was published in A. I. Chem. Eng., *Trans.*, v. 40, 1944, p. 317–331.

NOTES

95. RYDBERG INTERPOLATION TABLE.—In 1934 the Departments of Physics and Astronomy of Princeton University published at Princeton: *Rydberg Interpolation Table. Values of the function* $109737 \cdot 4/(n + \mu)^2$ for all values of $n + \mu$ from 1.000 to 11.000 in steps of .001. 24 p. 30.7 × 26.9 cm. It is a 4S to 6S table with differences, for $n = 1(1)10$, $\mu = 0(.001)1$. We are told that the original calculations were made by Miss JANET MACINNES and were checked by Dr. DONALD N. READ. There is no suggestion as to any earlier table of the kind (in the field of spectroscopy), as to who Rydberg was, or as to when or where the function here tabulated first appeared. On looking into the matter I found that the function was first given in K. Svensk. Vetensk.-Akad., *Handlingar*, v. 23, no. 11, 1890, p. 42, by JOHANNES ROBERT RYDBERG (1854–1919) who, at the time of his death, was professor of physics and director of the Physical Institute at the University of Lund. The memoir in question is entitled, "Recherches sur la constitution des spectres d'émission des éléments chimiques," and there is a 5S–6S table, p. 48–51, of $109721 \cdot 6/(n + \mu)^2$, for $n = 1(1)9$, $\mu = 0(.01)1$, Δ.

R. C. A.

QUERIES

28. MOSELEY ON DEFINITE INTEGRALS.—An article by HENRY MOSELEY (1801–1872) on "Definite Integrals" is reasonably accessible on p. 491–544 of *Enc. Metrop.*, Pure Sciences, v. 2, London, issues of 1843 and 1845. It reprints Legendre's values of K, E for $\theta = [0(1°)90°; 12D]$. It was first published, apparently separately, in 1837, and indeed the *Catalogue of the Crawford Library of the Royal Observatory*, Edinburgh, 1890, lists, under MOSELEY, *A Treatise on Definite Integrals*, London, 1837, 56 p. Where are other copies of this rare issue?

There was a later edition of the *Encyclopedia* in cabinet form with much smaller pages, and Moseley's article was advertised to appear in this also, but I cannot trace that it ever did. It is not in the British Museum set. Perhaps some reader can supply more definite information.

A. FLETCHER

QUERIES—REPLIES

37. HENDRIK ANJEMA (Q23, v. 2, p. 374).—The Archivist of the State Archives of our province of Friesland made a detailed search for information regarding this author of *Table of Divisors of all the Natural Numbers from 1 to 10000*, Leyden, 1767. As remarked in the Query this volume opens with an "Avertisement of the Booksellors": "After having taught with success & applause, for several years, the Mathematics in the University of Franequer, he resolved to devote the leisure hours . . . to the advantage of his old Disciples. . . ." Moreover L. CHERNAC writes on p. VIII of his *Cribrum Arithmeticum:* "Henricus Anjema, Lector Matheseos Franequeranus, in animo habebat separato volumine. . . ."

Now the Archivist informs me that all of this is pure fantasy on the part of "the Booksellors." Hendrik Anjema never taught at the University of Franeker. On May 9, 1740 Anjema was appointed as a *beadle* of that University and continued that employment up to 1750. But in June 1747, Professor N. Ypei (who was himself a "lector matheseos" before his appointment to a professorship) gave Hendrik Anjema "remisso examine et honorario," the titles "landmeter," "wijnroeier" and "ingenieur," that is, "landsurveyer," "wine-gauger" and "engineer." This agrees in some measure with our present "doctor honoris causa." That H. A. was born in Anjum (a small village in Friesland) is mentioned in a declaration by Prof. Ypei. On investigating the old records the Archivist was unable to determine in what year our Hendrik was baptized, since so many Hendriks are listed. In the record of his marriage to Magdalena Carolina Dronrijp at Harlingen on July 30, 1741 his name is given as Henricus Dominicus Anjema. In 1750 the "mathematicus" Anjema was appointed to the office of "clerk of victuals," while living at Lemmer, again a village in Friesland, and there he died, March 26, 1765. In a journal of that time is an advertisement of sale at Harlingen, Sept. 26, 1765, of Anjema's books, manuscripts and instruments (including a Gregorian telescope). The Archivist adds that H. A. is one of the numerous group in our northern provinces self educated

in mathematics and physics. The University of Franeker existed from July 29, 1585 to 1811.

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CORRIGENDA

V. 1, p. 217, I, D_1, 7, for Traub, $n = 0(1)19$, $s = 1(1)8$, *read* Traub, $j_{n,s}$, $n = 0(1)19$, $s = 1(1)8$, and $j'_{n,s}$, $n = 0(1)22$, $s = 1(1)9$

V. 1, p. 293, l. 16, *for* second, unchanged, reprint 1929, *read* second reprint 1928

V. 1, p. 330, l. −6, *for* u/k, *read* u/K.

V. 1, p. 446, l. 15, *for* 1(1)8, *read* 0(1)8

V. 2, p. 381, right-hand column, l. 3, *for* 389; II, *read* 389, 429; II.